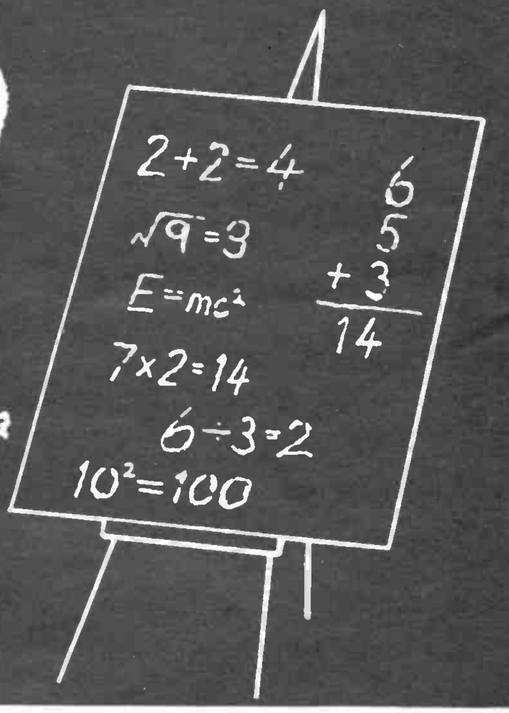
CALCS



by Jeff Scott Part 1

Complex Numbers

Were you bored by maths at school? Unfortunately, the teaching of maths, not only in schools but also in engineering colleges, tends to be a pure science instead of showing the practical applications to electronics and other branches of engineering.

Mathematics is an exact science with the answers popping out like a pudding out of a pudding bowl, as my teacher used to put it. The frontiers of mathematics, the philosophical side usually predicts the trend that engineering is likely to take.

For instance, the large scale digital processing of signals was shown mathematically long before the technology was available. Fourier analysis showed the components in a waveform much before spectrum analysers were available. Explosions and massive gravitational forces in space, compressing material into a black hole were explained mathematically before the physical phenomena were understood.

Mathematics also serves the purpose of manipulating data into the required form. For example, Laplace transforms transfer an expression from the frequency domain into the time domain. This is useful in electronics since old methods of analogue amplification and filtering were in the frequency domain (frequency division multiplex) whereas modern techniques are in the time domain (time division multiplex).

Complex numbers is an important branch of mathematics with applications to electronic engineering. Mathematicians use the letter 'i' in complex numbers, but 'j' is used in electronics since the letter 'i' in electronics is reserved for current. The 'j' symbol is often called the 'j' operator. The need for a 'j' operator arises for the following reasons. In a circuit that is purely resistive the current is in phase with the voltage, see Figure 1. In a circuit that is reactive, the current may lead or lag the voltage depending on whether the circuit is inductive or capacitive, see Figures 2 and 3.

Argand, a mathematician, was the first to invent the use of 'i' or the imaginary operator. This was useful for explaining the square roots of negative numbers. Consider the equation $x^2 - 16 = 0$. The solution is $x = \pm 4$ which is

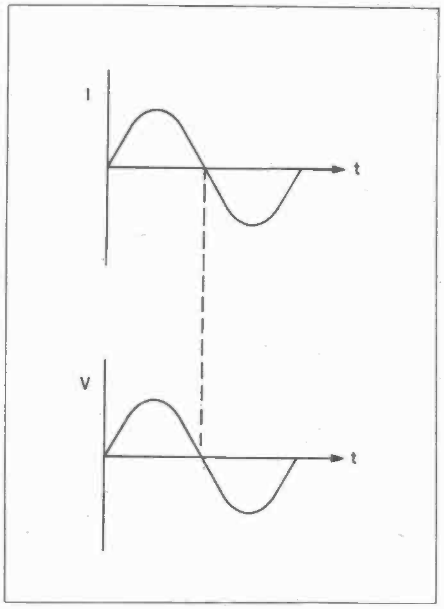


Figure 1. Current in phase with voltage.

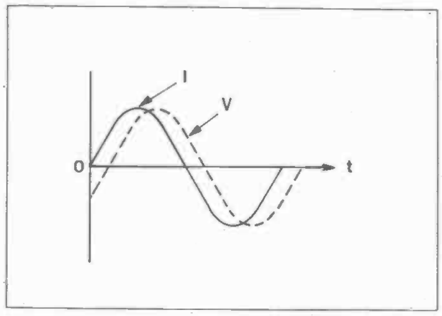


Figure 2. Current leading voltage.

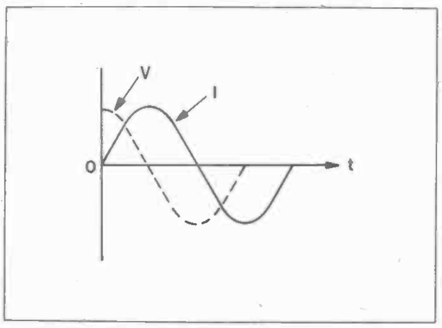


Figure 3. Current lagging voltage.

represented on an Argand diagram as in Figure 4.

But how does one solve the equation $x^2 + 16 = 0$?

$$\text{Here } x^2 = -16$$
$$x = \pm 4\sqrt{-1}$$

There is no real number which represents the square root of -1. Argand's solution to this problem was to assign real values along the x axis and imaginary values along the y axis. The solution of $x^2 = -16$ is now shown in Figure 5.

In electrical engineering there are no imaginary values but we call these quadrature

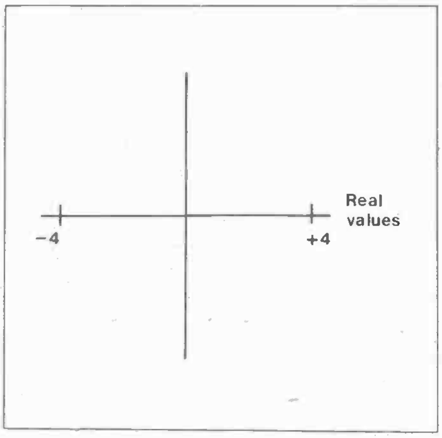


Figure 4. Argand diagram.

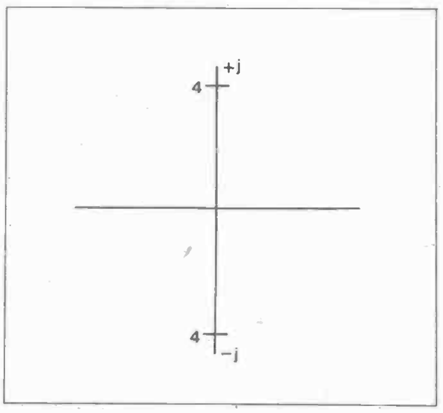


Figure 5. Imaginary (quadrature) components.

components. Therefore a purely resistive circuit will have real components only and a reactive circuit will have quadrature components. An Argand diagram has facilities for describing a circuit that has both real and quadrature components.

From now on we shall use the j symbol and examine the addition and subtraction of complex numbers. An expression that has both real and quadrature components is called a complex number. For instance a number that has 2 units of real value and 4 units of quadrature value would be represented as in Figure 6. The vector P is represented in the general form of: a + jb, or 2 + j4 in this case.

The above method of representing a complex number uses the Cartesian co-ordinates. There is another method of representing complex numbers using polar co-ordinates. In this method, if we define an angle θ (Figure 7) from a reference point then we have one of the attributes defined.

The other attribute must be the length of the vector. Both quantities can be calculated from the cartesian co-ordinates. From Pythagoras' theorem the hypoteneuse in Figure 7 is given by:

$$\sqrt{a^2 + b^2}$$

$$\text{or} \quad \sqrt{20^2 + 15^2}$$

$$= \sqrt{400 + 225}$$

$$= \sqrt{625}$$

$$= 25$$

The tangent of angle θ is $\frac{15}{20} = 0.75$

Therefore $\theta = 36.87°$

The resultant is sometimes called the modulus and the angle is called the argument.

Addition and Subtraction

One of the advantages of the complex notation is that numbers are added and subtracted quickly and easily. This is done by adding or subtracting the real components separately from the quadrature components. The alternative is the tedious method of drawing parallelograms and finding the resultant of only two vectors at a time.

Figure 8 (not to scale) shows three complex numbers 2 + j2, −5 + j8, −10 − j4. Adding the real components gives −13 and adding the quadrature components produces j6. Therefore the answer is −13 + j6. Subtraction is just as easy compared to the parallelogram method of operating on only two vectors each time.

Multiplication and Division of Complex Numbers

Before we deal with the multiplication of complex numbers we must see what happens when we multiply j by itself successively. Since:

$$j = \sqrt{-1}$$

$$j^2 = -1$$

$$j^3 = -1 \times j = -j$$

$$j^4 = -1 \times -1 = +1$$

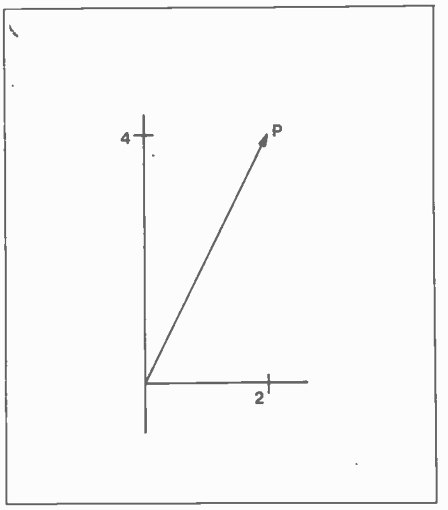


Figure 6. Cartesian co-ordinates.

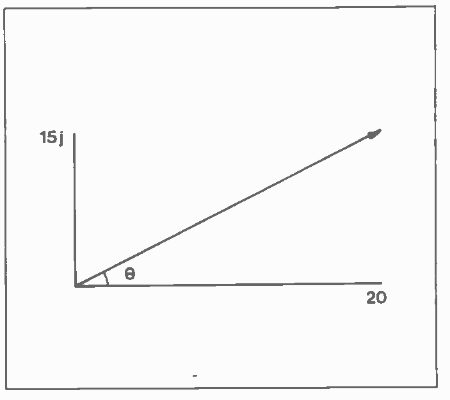


Figure 7. Polar co-ordinates.

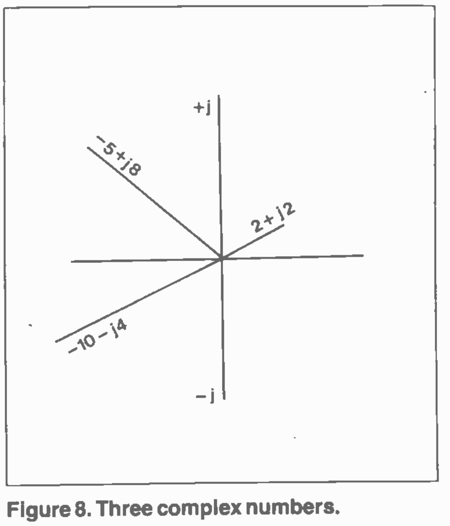


Figure 8. Three complex numbers.

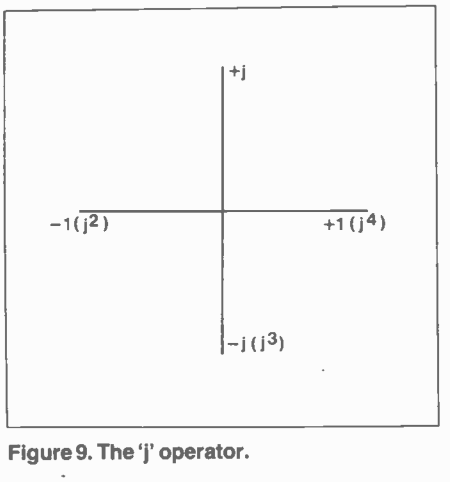


Figure 9. The 'j' operator.

Now we can see why it is called an operator. From Figure 9, each successful multiplication by j has the effect of swinging the value through 90 degrees.

In order to multiply two complex numbers together, we simply expand the brackets in the usual manner.

For example: $(3 + j^5)(4 - j^2)$

$$12 + j20 - j^2 2$$

Remembering $j^2 = -1$,

$$12 + j20 + 2$$

$$14 + j20 \text{ is the result.}$$

Division of complex numbers is just as simple, the answer dropping out like a pudding out of a pudding bowl.

For instance $\frac{3 + j5}{4 - j2}$

First we rationalise the denominator by multiplying it by its conjugate. The conjugate has the same values but opposite phase and has the effect of turning into a wholly real number. The conjugate of 4 − j2 is 4 + j2.

$$\text{So} \quad \frac{(3 + j5)(4 + j2)}{(4 - j2)(4 + j2)}$$

$$= \frac{12 + j6 + j20 + j^2 10}{16 + j8 - j8 - j^2 4}$$

$$= \frac{12 + j26 - 10}{16 + 14}$$

$$= \frac{2 + j26}{20}$$

$$= 0.1 + j1.3$$

Multiplication and Division using Polar Co-ordinates

Polar co-ordinates lend themselves quite easily to multiplication and division. The modulus is multiplied separately and the angles (argument) are added together in multiplication:

$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1 \times r_2\angle\theta_1 + \theta_2$$

$$6\angle 35 \times 7\angle 40 = 42\angle 75$$

For division the moduli are divided and the angles are subtracted:

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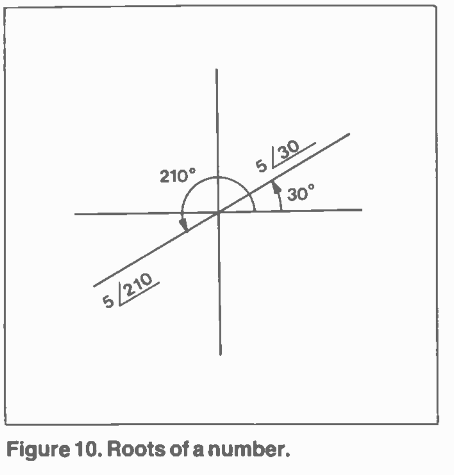

Figure 10. Roots of a number.

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$$r_1\angle\theta_1 \div r_2\angle\theta_2 = \frac{r_1}{r_2}\angle\theta_1 - \theta_2$$

$$25\angle 60 \div 5\angle 40 = 5\angle 20$$

Polar co-ordinates are particularly useful in finding square roots. The square root of the modulus is found and the angle divided in half:

$$(r\angle\theta)^{1/2} = \sqrt{r}\angle\theta/2$$

$$(25\angle 60)^{1/2} = 5\angle 30$$

Now a real number like 25 has two roots 5 and −5.

These are of the same magnitude but opposite sign (phase) to each other. Similarly, a complex number has two roots, one 180 degrees out of phase (opposite phase) to each other.

Hence $(25\angle 60)^{1/2}$ has roots of $5\angle 30$ and $5\angle 210$, see Figure 10. To square a number in polar co-ordinates we square the modulus and double the argument.

$$(r\angle\theta)^2 = r^2\angle\theta \times 2$$

$$(4\angle 20)^2 = 16\angle 40$$

Application of Complex Numbers to A.C. Bridges

We shall now see how many of the above methods are applied to the solution of equations for AC bridges. In an equation containing both real and imaginary (quadrature) terms, the real components can be equated separately from the quadrature components.

For instance:

$$R + jwL = 13 + j19$$

Therefore $R = 13$

and $jwL = j19$

Let us examine a more complicated case like the Maxwell bridge of Figure 11.

At balance $Z_1 Z_x = Z_2 Z_3$

where $Z_1 = R_1 \dfrac{(1)}{(jwC_1)} = \dfrac{R_1}{jwC_1R_1 + 1}$

$Z_2 = R_2$
$Z_3 = R_3$
$Z_x = R_x + jwL_x$

Substituting in $Z_1 Z_x = Z_2 Z_3$:

$$\frac{(R_1)}{(jwC_1R_1 + 1)}(R_x + jwL_x) = R_2R_3$$

Multiplying both sides of the equation by $(jwC_1R_1 + 1)$:

$$R_1(R_x + jwL_x) = R_2R_3(jwC_1R_1 + 1)$$

$$R_1R_x + jwL_x = jwC_1R_1R_2R_3 + R_2R_3$$

Equating real terms:

$$R_1R_x = R_2R_3$$

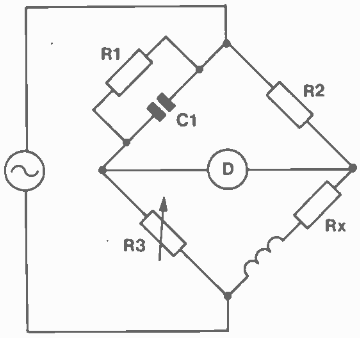

Figure 11. Maxwell bridge.

$$R_x = \frac{R_2R_3}{R_1}$$

Equating imaginary terms:

$$wL_x = wC_1R_1R_2R_3$$

$$L_x = C_1R_1R_2R_3$$

This enables one to calculate the value of an unknown inductor L_x and its associated resistance R_x from known values of the other components in the bridge. It is hoped that the above shows how even large mathematical problems can be tackled in small easy steps.