Designer's Notebook

Audio Design
Clearing a path through the complex numbers used in audio design.

By John Linsley Hood

ONE of the features of audio circuitry, with the partial exception of audio power amplifiers which are largely flat frequency response devices, is that some modification of the gain/frequency characteristic is needed to correct for uneven recording or replay frequency responses, or to emphasise or exclude desirable or unwanted parts of the frequency spectrum. This is done by inserting a combination of resistors and capacitors (or inductors) in the signal path, or, possibly, in the feedback path around an amplifier. This is a very powerful technique, and with sufficient ingenuity in the circuit design, all sorts of shapes of frequency response can be achieved. However, it requires the ability to do reasonably accurate calculations of systems using capacitors or inductors in combination with resistors, and this immediately runs into the problem of the phase shifts which occur within such networks. I will explain.

If one passes an alternating current through a series combination of a resistor and a capacitor or a resistor and an inductor, the voltages developed across the two components will be 90° out of phase with each other. I have shown this graphically in Fig. 1a and 1b. Also, while the voltage developed across a capacitor will 'lag' in phase in relation to the current flowing through it, (because the voltage across a capacitor depends on the charge within it and it takes time for the capacitor to charge up or discharge), the opposite is true of an inductor, in which the voltage will 'lead' in phase with reference to the current (due to the instantaneous generation of a 'back EMF' in an inductor which seeks to oppose any change in current).

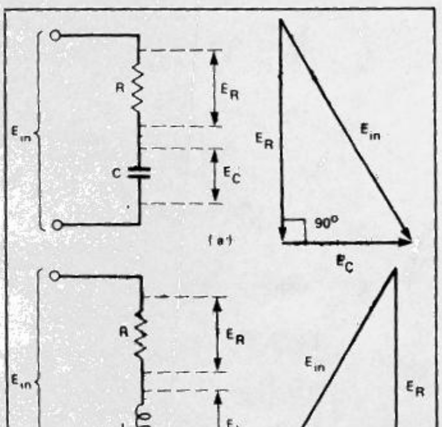


Fig. 1 Phase angle relationships in RC and RL networks.

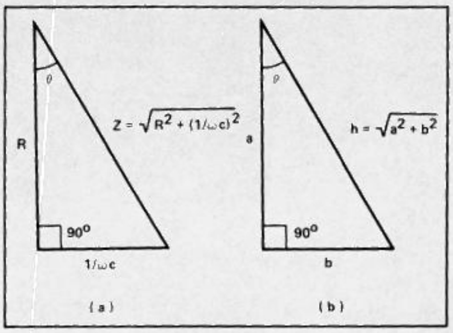


Fig. 2 Impdeance diagram for an RC network.

We have seen earlier in this series that the impedance of a capacitor (Zc) is related to its capacitance and the operating frequency by the equation $Z_c = 1/2pifC$. Similarly, the impedance of an inductor $Z_L = 2pifL$, where f is the frequency and C and L are in Farads and Henries respectively. Because of the effects of phase shifts, any calculation we made, say, of the attenuation of an RC or LC network based on these formulae for impedance would probably give incorrect answers. We therefore need a better method.

The j Symbol

There is, conveniently, a mathematical trick which enables us to do calculations which take into account the phase shifts produced by inductors and capacitors, and this is the operator i or j, which is numerically the square-root of -1. Pure mathematicians call this i to denote the fact that it is an imaginary number, since all real numbers give positive values when they are squared. However, since electrical engineers have already adopted the symbol i to denote electrical current, we refer to the square root of -1 as j instead. The use of this j operator is not as ridiculous as it might seem, as a way of describing a 90° phase shift, for the following reason.

In DC systems, the opposite of a positive voltage + V is a negative voltage -V. In an AC system, the opposite of an instantaneous positive potential (and it is convenient to refer to such AC potentials as E to distinguish them from DC voltage ± V) is the same potential half a cycle (180°) later when it has swung from positive to negative. A 180° phase shift in an AC signal therefore has the effect of multiplying the potential by -1, provided always that the signal we are talking about is sinusodial.

Now, if we have two RC (or LC) networks in series, both of which produce a 90° phase shift (and two such networks in series will have a multiplying effect on the signal just as ½ x ½ = ¼), the final effect is a 180° phase shift (= x -1). If we want to represent these phase shifts mathematically, we must find something which, when multiplied by itself gives the result -1. The square-root of -1 is just such a thing. It can therefore be used in our calculations as a way of denoting 90° phase shift.

The other bit of shorthand which circuit engineers normally use in these calculations is Greek symbol Omega which appears here as *w* to denote 2*pi*f, since these terms nearly always occur together. The true impedance of a capacitor or inductor is, therefore, not $Z_c = 1/2pifC$ or $Z_L = 2pifL$, but $Z_c = 1/j2pifC$, and $Z_L = j2pifL$. In shorthand form this becomes $Z_c = 1/jwC$ and $Z_L = jwL$.

Since the phase shift produced by the L or C elements in RC or LC networks is 90°, we can represent the behaviour of this circuitry in a graphical form as shown in Fig. 1, as a right angled triangle, where the "j" term denotes the right angled limb, and this allows us to derive some further bits of information. Taking the case of a simple RC series network, as in Fig. 1a, the circuit impedances can be represented as in Fig. 2a, in which the vertical and horizontal limbs represent the resistive and capacitative impedances R and 1/j*w*C respectively. It is unnecessary to write the "j" symbol in the capacitance impedance limb of the drawing; that is implicit in its position at right angles to the R limb. From the theorem of Pythagoras, the length of the hypotenuse, h in Fig 2b, is the square-root of $a^2 - b^2$, and from fairly simple trigonometry, the angle $\theta = Tan^{-1} b/a$, a calculation which a lot of pocket calculators will do very quickly.

Returning to our impedance diagram of Fig 2a, the resultant impedance of our network is therefore

$$\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

We can also determine the phase angle, θ, between the voltage developed across the network and the current flowing through it which will lag by θ, which is $\mathrm{Tan}^{-1}\ 1/wCR$. (If C were very large indeed, or R were very large, the phase shift would be nearly zero.)

To recapitulate, we can identify the phase shifting characteristics of Cs and Ls by coupling the symbol j to their impedance equations, and we can derive the resultant impedance and phase angle of these 'complex' networks by sorting out the terms with and without the j symbols, and using them in simple geometric or trigonometric calculations. This process holds good no matter how many Rs, Cs and Ls we have in our network, it just becomes more complicated if there are more phase shifting elements.

The thing, however, which we must watch, is that we keep the real and the imaginary (j containing) parts separate in the final equation at which we arrive. Now let us look at some real life examples.

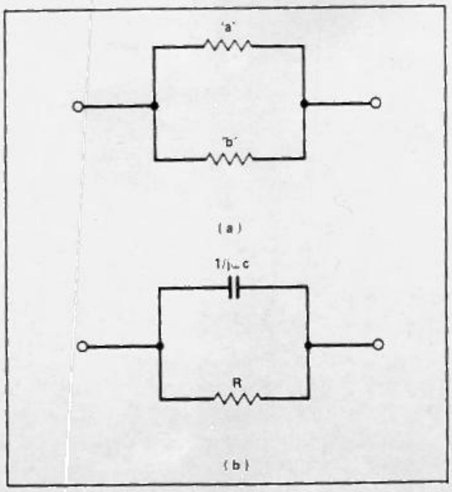


Fig. 3 Impedance of an RC parallel network.

Impedance Of RC Parallel Network

If the components were a and b as in Fig 3a, their impedance, when in parallel, would be

$$\frac{ab}{a+b}$$

Therefore, if they are R and $1/jwC$, as in Fig. 3b, their parallel impedance will be

$$Z = \frac{(1/j\omega C).\ R}{1/j\omega C + R}$$

if we multiply the top and bottom of this equation by jwC, we can get it into the much more manageable form

$$Z = \frac{R}{1 + j\omega CR}$$

The next mathematical dodge is to get rid of the js in the bottom line of this equation, so that we can divide it up into two separate parts, one without js and one with them representing the in-phase and the 90° 'quadrature' components.

This can be done by using the relationship

$$(a + b)\ (b - b) = a^2 - b^2.$$

If it was $(a + jb)\ (a - jb)$ the result would be $a^2 + b^2$, bearing in mind that $j^2 = +1$. The important thing is that j terms have disappeared. We can, therefore, multiply the top and the bottom of an equation containing a j term in the bottom line by a $- jb$ and eliminate these terms from the denominator leaving two separate fractions, which meets our original requirement for a usable equation. Treating the

$$Z = \frac{R}{1 + j\omega CR}$$

equation like this, we end up with

$$Z = \frac{R}{1 + (\omega CR)^2} - \frac{j\omega CR^2}{1 + (\omega CR)^2}$$

which allows us to calculate both the impedance and the phase angle between current flow and voltage, in our CR parallel network.

Attenuation Of An RC Network

The circuit shown in Fig. 4b is a very versatile one in that, as it stands, it is a useful 'step' attenuator network, while if R2 = 0 it is a simple HF attenuator circuit. Looking at the resistor network of Fig 4a, the attenuation of this would be

$$\frac{Eout}{Ein} = \frac{Rb + Rc}{Ra + Rb + Rc}$$

By analogy, therefore, the performance of Fig. 4b will be

$$\frac{Eout}{Ein} = \frac{1/j\omega C + R2}{R1 + 1/j\omega C + R2}$$

and this can be simplified to

$$\frac{Eout}{Ein} = \frac{1 + j\omega CR2}{1 + j\omega C\,(R1 + R2)}$$

by multiplying top and bottom of JwC. Doing the necessary mathematical manipulation extracts the in-phase and quadrature components as

$$\frac{Ein}{Eout} = \frac{1 + \omega^2C^2R2\,(R1 + R2)}{1 + \omega^2C^2\,(R1 + R2)^2} - \frac{j\omega CR1}{1 + \omega^2C^2\,(R1 + R2)^2}$$

and if we make R2 = 0, the right hand side of this equation simplifies to

$$\frac{1}{1 + \omega^2C^2R1^2} - \frac{j\omega CR1}{1 + \omega^2C^2R^21}$$

In this case also we have separated out the in-phase and quadrature components, so that the transmission factor is obtained by doing a square-root of the sum of the squares of these, and the phase angle of the output is given by

$$\mathrm{Tan}-1\left(\frac{\text{quadrature}}{\text{in-phase}}\right)$$

It is always useful, when one comes to the end of an algebraic manipulation like this, to check that one hasn't done anything wildly silly by putting in some limit values. For example, in the equations above, consider the effects of C = 0. This causes the equation to become

$$\frac{Eout}{Ein} = 1$$

which is what we would expect, (assuming the load is infinitely high in resistance). On the other hand, if C is extremely large, the first example gives

$$\frac{Eout}{Ein} = \frac{R2}{R1 + R2}$$

and the second gives

$$\frac{Eout}{Ein} = 0$$

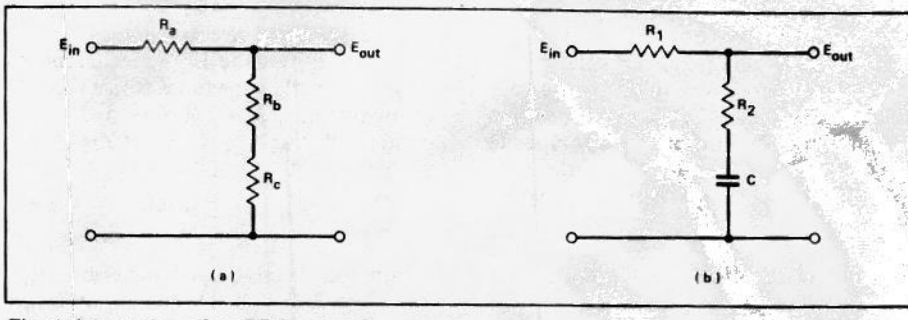
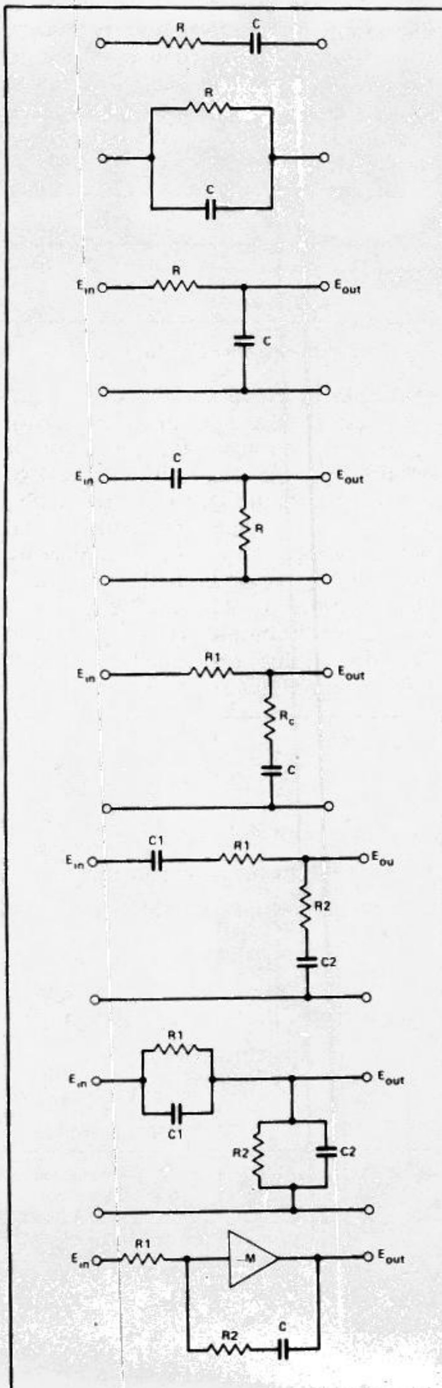
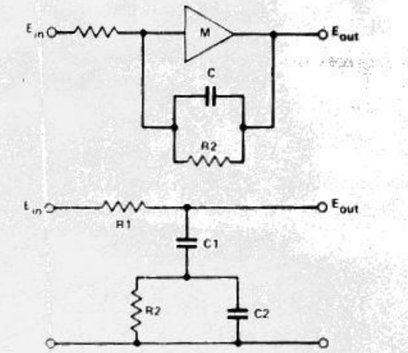


Fig. 4 Attenuation of an RRC network.





$$\frac{Eout}{Ein} = \frac{1}{1 + j\omega CR}$$

$$\frac{Eout}{Ein} = \frac{j\omega CR}{1 + j\omega CR}$$

$$\frac{Eout}{Ein} = \frac{1 + j\omega CR2}{1 + j\omega C\,(R1 + R2)}$$

$$\frac{Eout}{Ein} = \frac{C1 + j\omega C1C2R2}{C1 + C2 + j\omega C2\,(R1 + R2)}$$

$$\frac{Eout}{Ein} = \frac{C1 + j\omega C1R1R2}{R1 + R2 + j\omega R1R2\,(C1 + C2)}$$

$$\mathrm{Gain} = -M\,\frac{1 + j\omega CR2}{1 + j\omega C\,(R2 + MR1)}$$

$$\mathrm{Gain} = -M\,\frac{R2}{R2 + MR1\,(1 + j\omega CR2)}$$

$$\frac{Eout}{Ein} = \frac{1 - \omega R1R2C1C2 + j\omega\,(R1C1 + R2C1 + R2C2}{1 + j\omega R2\,(C1 + C2)}$$

Fig. 5 Characteristics of some common RC networks.

Modern programmable pocket calculators make the task of calculating the characteristics of such RC networks relatively easy, once the labour of working out the maths has been done, and although I haven't shown any yet, the process of calculation in RL networks is very similar. One can then, for example, write a suitable programme with the component values held in the calculator memory, and let the calculator go through the process for any frequency value which one enters before pressing the run button.

To remove some of the labour in calculation I am showing in the composite Fig. 5 a selection of RC networks with their impedance and transmission equations.

Resistor-Inductor Networks

The method of calculating the performance of these is identical to that for RC networks, except that one uses jwL instead of 1/jwC in the equations. For example, the circuits of Fig. 6a and 6b have transmissions

$$\frac{E_{out}}{E_{in}} = \frac{j\omega L}{R + j\omega L} \text{ and } \frac{R}{R + j\omega L}$$

respectively, which can be broken down into the in-phase and quadrature components as

$$\frac{(\omega L)^2}{R^2 + (\omega L)^2} + \frac{j\omega LR}{R^2 + (\omega L)^2}$$

and

$$\frac{R^2}{R^2 + (\omega L)^2} - \frac{j\omega LR}{R^2 + (\omega L)^2}$$

In all the equations shown, it is possible (as I am sure you will have spotted) to change one kind of network into a simpler one by putting value of R or C or L equal to 0. As an example, if we make network (7) of Fig 5 have values of 0 for C1 and C2,

$$\frac{E_{out}}{E_{in}} = \frac{R2}{(R1 + R2)}$$

which is what we would expect. Or, by just deleting C1 (C1 = 0) we will end up with the equation of a type 3 network, when there is a resistor across the output.

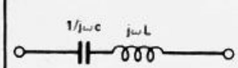
Some Practical Examples

A lot of the above may have been a bit dull reading for the non-mathematically inclined (which, I suspect, is 99% of us) and may tempt the reader to ask 'Well, that's all very nice, but what real use is it'. So I propose to show a few examples where there are some slightly surprising outcomes from the calculations.

(1) The LC series circuit.
Let us take first the LC series circuit of Fig. 7. Now it's impedance is just the sum of the two bits, Z = 1/jwC + jwL. If we multiply through by 1 (=jwC/jwC), we get

$$Z = \frac{1 - \omega^2 LC}{j\omega C}$$

This has an interesting characteristic, that if $w^2LC = 1$, Z = 0. This condition is met if $w^2 = 1/LC$ or $w = 2pi$ square of LC. So, at resonance, this series LC network looks like a short circuit. Away from resonance, there is a quadrature component due to the jwC term in the bottom line, which causes the phase of the transmitted signal to swing from + to − as the input passes through resonance.



$$\text{at } f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$Z = 0$$

Fig. 7 LC series resonant circuit.

(2) The Wien network.
This interesting and useful circuit, shown in Fig. 8, and the basis for a lot of oscillator designs is basically a network of the type shown in Fig. 5 (1) in series with one of the 5(2) type, with both Cs and both Rs being of the same value. Since we have already worked out the impedance characteristics of 5(1) and 5(2), we can write down the output, as a proportion of the input using the familiar a/(a+b) form, where 5(2) is a,

$$\text{at } f_0 = \frac{1}{2\pi\sqrt{CR}}$$

Eout = ⅓ Ein

with no phase shift.

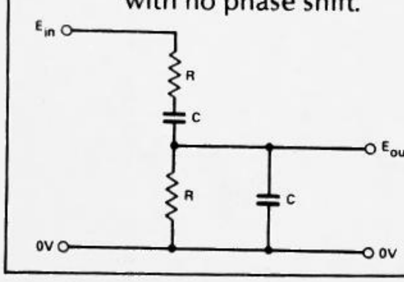


Fig. 8 The Wien network.

and 5(1) is b.

This gives the rather unwieldy looking equations

$$\frac{E_{out}}{E_{in}} = \frac{\frac{R}{1 + j\omega CR}}{\frac{R}{1 + j\omega CR} + \frac{1 + j\omega CR}{j\omega C}}$$

$$= \frac{\frac{j\omega CR}{1 + j\omega CR}}{\frac{j\omega CR}{1 + j\omega CR} + 1 + j\omega CR}$$

fortunately, this simplifies to:-

$$\frac{E_{out}}{E_{in}} = \frac{j\omega CR}{1 - (\omega CR)^2 + 3j\omega CR}$$

when $(wCR)^2$ = 1 'or wCR = 1, since (square root of 1 = 1)' this becomes,

$$\frac{E_{out}}{E_{in}} = \frac{j\omega CR}{3j\omega CR} = \frac{1}{3}$$

with no 'j' terms left. Now wCR (=2pifCR) = 1 when f = 1/(2piCR), which gives the frequency at which the Wien network output is in phase with the input, and has a magnitude of 1/3 that of Ein.

(3) The Sallen and Key active filter.
This is one of the archetypes of the class of circuit known as active filters, and is valuable because it can be built with a single op-amp in the form shown in Fig. 9a or 9b. These are high-pass and low-pass versions of the filter. The behaviour of the circuit is such that the gain is subtaintially level (and x1) at frequencies above, or below, some critical turnouver frequency — depending upon whether we are using a high-pass or low-pass arrangement — but beyond this frequency the gain falls at −12dB/octave, as shown in 9c and 9d. If we substitute impedance 'blocks' for the Rs and Cs, as shown in 9e, we can work out a model for the analysis of the circuit using the 'j' techniques described above. However, to simplify your calculations we will assume that our amplifier is an ideal one with unity gain, and has an infinitely high input impedance and a negligibly low output impedance.

We can derive the following relationships.

$E_{in} = E_{out} + (i_1 + i_2)Z1 + i_2Z2$. .... (1)

and $E_{out} = i_2Z4$ therefore $i_2 = E_{out}/Z4$ .... (2)

also il=(Ex−Eout)/Z3 and $(Ex+E_{out}) = i_2Z2$

Therefore $il = i_2Z2/Z3$. .... (3)
From (1) and (3)

$E_{in} = E_{out} + i_2Z1\,Z2/Z3 + i_2Z1 + i_2Z2$
.... (4)

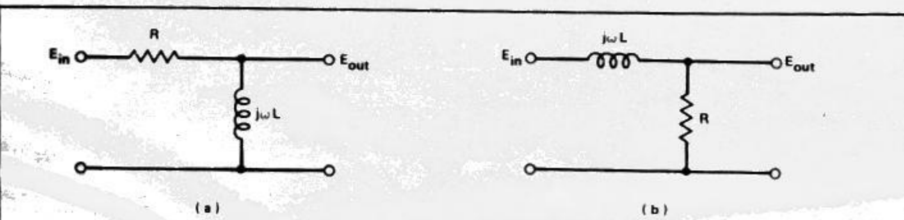


Fig. 6 RL networks.

and from (4) and (2)

$$Ein = Eout\,(1+Z1Z2/Z3Z4+Z1/Z4+Z2/Z4) \;.....\;(5)$$

Therefore

$$\frac{Ein}{Eout} = \frac{1}{1+\frac{Z1}{Z4}+\frac{Z2}{Z4}+\frac{Z1Z2}{Z3Z4}} = \frac{Z3Z4}{Z3Z4+Z1Z3+Z2Z3+Z1Z2} \;.\;(6)$$

We can now fit in the Rs and 1/jwCs in place of the Zs, and get the formulae for the real circuits. In the case of the low-pass filter, (9b and 9d), where Z1 = R1, Z2 = R2 and Z3 = 1/jwC1 and Z4 = 1/jwC2,

$$\frac{Eout}{Ein} = \frac{1}{1+j\omega C2(R1+R2)-\omega^2(C1C2R1R2)} \quad (7)$$

Several things can be deduced from this: where f = 0 (w = 0) the output is 1/1 (unity gain at VLF), where w^2(C1 C2 R1 R2) = 1 the denominator is at its smallest, and the output is therefore at a maximum. This is the turn-over frequency where f = 1/2*pi* square root of R1 R2 C1 C2, and at this point the output of the circuit is 1/jwC2 (R1 + R2), which can call the 'Q' of the circuit.

There is one further small trick which can be done with this calculation. Suppose we say that x = R1/R2 and y = C1/C2, then R1 = xR2 and C1 = yC2, and suppose that we call the frequency at which w^2(C1 Cr R1 R2) = 1, w_o, then $w_o{}'$ = 1/xy(C2R2)^2 and w_o = 1/C2 R2*pi*xy. Also, our middle term jwC2(R1 + R2) becomes jwC2R2(1 + x).

Let us now express our equation for frequncy as a fraction of w_o, the turn-over frequency, we can find that . . .(7) becomes,

$$\frac{Eout}{Ein} = \frac{1}{1+j\frac{\omega}{\omega 0}\frac{(1+x)}{\sqrt{xy}}-\left(\frac{\omega}{\omega 0}\right)^2}$$

and the 'Q', or gain at f_0, (when $\omega=\omega_0$) $\frac{\sqrt{xy}}{1+x}$

This gives us a means of calculating the performance of this filter circuit over a range of frequencies, of determining what its turn-over frequency will be, and of predicting the circuit Q at that frequency (for an optimally flat response from a 2 element filter of this type, Q should be 1/square root of 2 or 0.707).

I have only gone through the calculations for a low-pass filter in this instance, but the high pass version will follow if appropriate R2 and Cs are put in place of the Zs.

Conclusions

The use of the "j" operator, to simulate mathematically the effect of the phase shift in an inductor or capacitor allows useful and instructive calculations to be made on networks which contain Ls and Cs as well as resistances. With a programmable calculator, to take the labour out of the repetitive calculations, it becomes practical to calculate a frequency response — and phase shift — for any network which one has the patience to work out. This then, should allow us to explore the performance of our circuitry, while it is still at the 'drawing on paper' stage, and thus avoid surprises! ■

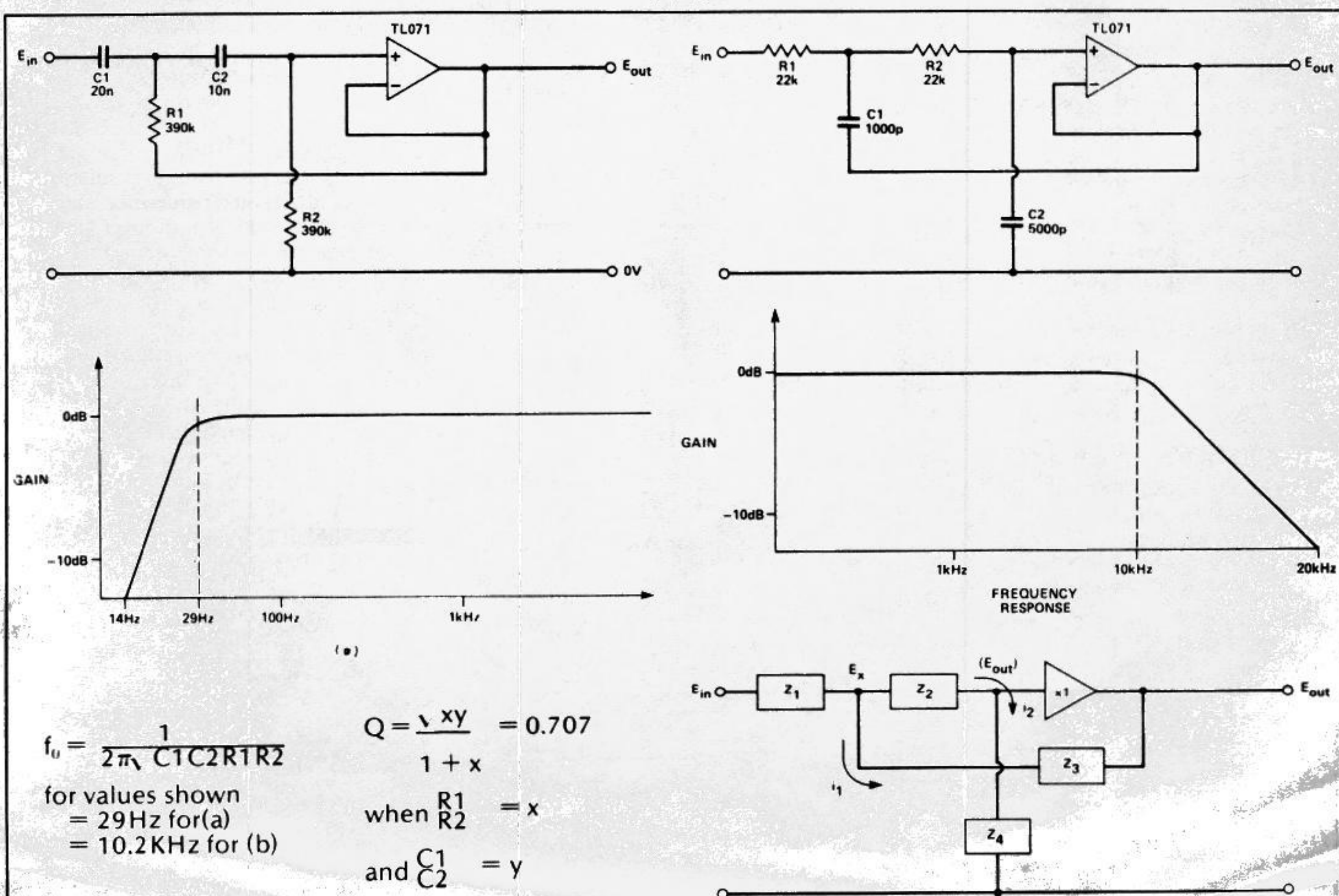


Fig. 9 Sallen and Key type active filters.