THE THEVENIN-NORTON STORY

Always been a little unsure of the circuit theorems of Thevenin and Norton? Here's an easy to understand explanation . . .

by BRYAN MAHER

Once upon a time (Oh no, he's going to tell another fairy story!) when a certain electronics lecturer had to go off to have his tonsils out, a young, overconfident know-it-all student named Harry offered to act as Apprentice Tutor teaching a class of avid students, the youngest of whom were a pair of twins named Sue and Fred. Harry was tired of spending all his nights devising laboratory experiments, only to find the students could perform them in thirty minutes flat. He decided to really challenge them.

Purchasing a lot of black boxes, one for each student, Harry had the college technician build into each box a copy of a circuit we show here as Fig.1. As you can see, his circuit consisted of six batteries (voltage sources), four current sources and twenty seven resistors in a complex pattern. A multi-pole switch was also wired in so that if the user wished she/he could switch off all the power in such a way that:
(a) all voltage sources were removed and their position shorted out; and (b) all current sources were removed and their position simply left open.

The Task

The complete circuit was concealed within the box, the only access to the circuit from the outside world being via the two terminals shown. The task young Harry presented to the students was simply:

"From any measurements you care to make at those two terminals on the outside of the box, can you tell me what circuit is inside the box? The box is sealed, you cannot look inside but all circuits are the same, all voltage and current sources are ideal or near enough".

The students laboured long and hard, racking their brains for a week. All except the twins, Sue and Fred, who simply took their boxes home where each made two measurements:
1. With the voltage and current sources switched on, they measured the output voltage at the box terminals when no external load was connected thereto. They called this V1.
2. With all sources switched off (voltage sources replaced by a short circuit, current sources replaced by an open circuit) they used an external ohmmeter to measure the equivalent resistance at those two terminals. They called this "the resistance looking back into the box" and named this value R1.

Fred then quietly purchased another box which looked identical, fitted a switch and two terminals, but inside his box he installed an extremely simple circuit, with just one battery as a voltage source and one resistor in series with it, as in Fig.2.

Fred chose his battery of the same voltage as the previously measured V1, and chose his one resistor the same as the previously measured value R1.

Sue, to do something more interesting still, also purchased an identical box fitted with the switch and two terminals, but built a different circuit inside her

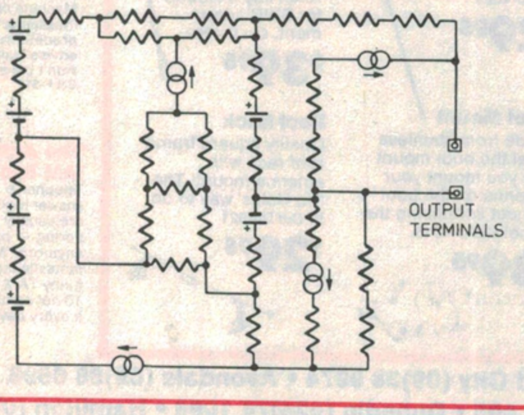

Fig.1: Harry's foolish circuit. Current sources are shown as linked circles, while switches are not shown for clarity.

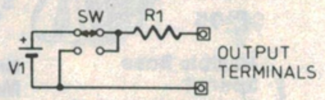

Fig.2: Fred's equivalent circuit (Thevenin Equivalent).

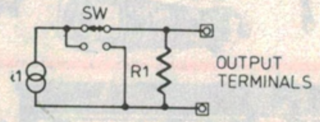

Fig.3: Sue's equivalent circuit (Norton Equivalent).

box. She used the same value resistor R1, but connected it straight across the two terminals, also connecting a current source in parallel with the same two terminals as in Fig.3.

A few minutes quiet reflection, and she had decided what value of current her current source i1 should be. To do this she simply used Fred's values of V1 and R1, and calculated what current would flow if Fred were to short-circuit his box's terminals. This current is obviously given by

i1 = V1/R1

Using that value i1 for her current source, she assembled her circuit and box.

Each putting a secret mark for identification of their boxes, the twins threw the two original boxes made in the lab into the rubbish can, quietly returned to college and placed their own boxes with all the others in the lab.

Three measurements

Next day in Laboratory class the "apprentice-tutor", believing he had outwitted his students, offered to open one box in front of the class and show them the thirty seven components. When asked were all the circuits the same he replied "yes" and proceeded to demonstrate all possible external measurements which could be made at the two terminals.

He could think of three possible tests thus:

1. With all sources switched off (all voltage sources removed from circuit and replaced by a short; all current sources removed from circuit and their position left open) he applied an ohmmeter to the output terminals and noted the reading. This test he applied to every box, with the same result.
2. With all sources switched on he measured the output voltage at the terminals with no external load connected thereto. Repeating his test on every box, he showed that they all read the same.
3. With all sources switched on, he short-circuited the output terminals via a current meter and noted the value of short circuit current. Quickly he demonstrated that every box gave the same result.

The apprentice-tutor then claimed his tests were sufficient to show that all boxes contained the same circuit, that complex collection of thirty seven components, Fig.1. He (with a smirk of triumph) now proudly displayed this circuit diagram to all students.

Feeling a little put down, a few students asked to see the boxes opened, wishing to see this fantastic complex circuit which had defeated them. The apprentice tutor complied.

Choosing a box at random, his knowing smile broadened as he proceeded to open it to display the contents. He would enjoy showing them his complex circuit.

Surprise

But his face changed to ghost-like bewilderment at what he saw! For as luck would have it he had picked up Fred's box — empty except for one battery and one resistor!!! In panic, fearing some trick, he grabbed another box, opened it quickly and lo-and-behold there was nothing in Sue's box but one resistor and one current source!

In a frenzy now, the tutor opened a third box, relieved at last to find his complex circuit of 37 components. But he was now really between a rock and a hard place! He had to explain how all boxes gave identical readings in all possible measurements that could be made from outside!

Desperately calling "time out", Harry raced home to bury his nose in his favourite electronics text book to find the answer, realising that he must have missed something in his own education. Could it have been one of those days he "wagged it" to go sailing?

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There in his text book for all to see was the theorem which Fred had discovered. Harry read:

"Thevenin's Theorem: Any DC circuit at all (think of Fig.1) **connected to two terminals, can be regarded as equivalent (in its effect on external circuits connected to those two terminals) to a different circuit consisting of one voltage source V1 and one resistance R1 in series** (think of Fig.2) **provided that:**

1. V1 is made the same as the voltage which would be measured at those terminals if all external connections were removed.

2. R1 is made the same as that resistance which would be measured looking back into the circuit from the two terminals, with all external connections removed from those terminals, and all sources reduced to zero (all voltage sources off and shorted out, all current sources off and left open circuit).

3. The above two independent measurements/calculations are sufficient. However, as an option, a third measurement/calculation is possible. If the two terminals were short circuited while all sources are switched on, either calculate (or measure, if safe) the short-circuit current i1 which would flow if the two terminals were shorted. (Caution! Do not actually short those terminals if danger or damage will result!!). Such a calculated (or measured) short-circuit current makes a third item of information. The three measurements form a dependent set, i.e., only any two are needed, from any two the third can readily be calculated, as they are related by

R1 = V1 / i1

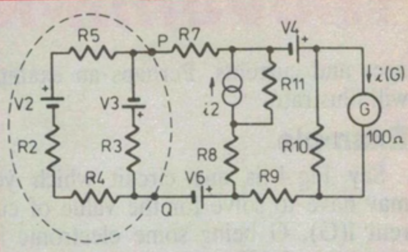

Fig.4: A real circuit example of Thevenin-Norton Equivalent reductions. The task here is to calculate i(G) given the values of all components and voltage sources, and the current source i2. P and Q are two arbitarily chosen points so chosen to enclose a simple circuit section within the dotted loop. The text explains.

So that, thought Harry, explains what Fred had done. Now what about Sue's circuit? No voltage sources, only a current source and one resistor (Fig.3). Soon Harry found in his trusty text book Norton's Theorem which reads:

Nortons Theorem: Any circuit at all (for example Fig.1) **connected to two terminals can be regarded as equivalent (in its effect on any external circuit connected to those two terminals) to a different, very simple circuit consisting of one current source i and one resistor R1 both in parallel with those two terminals** (as Fig.3) **provided that the three conditions (as given above for Thevenin's theorem) are met.**

Impossible task

Harry realised what Sue had done — all was now revealed to him. Clearly the task he had set those students was impossible for, as Sue and Fred had demonstrated, the three dependent measurements listed above are the only ones possible and they will never establish the actual details of a circuit connected to those terminals — only the *equipment* circuit.

You gentle reader, may wonder if this story is of any use to you? Yes, oh yes! It is of great benefit to all who seek to solve circuits, i.e., find values of volt-

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ages and currents. Perhaps an example will illustrate.

Example

Say Fig.4 is any circuit which you may have to solve for the value of current i(G), G being some electronic instrument of 100 ohms impedance. You can, in your imagination, nominate any two points P and Q in the circuit, call them two "terminals" and through them draw a closed ring around any parts of the circuit you wish, as the dotted line in Fig.4.

The only rules are that: (i) the dotted ring drawn does not cut any wire or component, but must cut through both points P and Q and (ii) components within the dotted ring must be linear, which implies constancy and reciprocity (ie end-for-end-ability).

Now you can call that dotted ring your black box, the points P and Q the imaginary "terminals", and you can imagine everything outside that ring to be removed. By relatively simple calculations you can calculate the values of V1, R1 and i1 which would be measured at "terminals" P and Q if the dotted ring and the components within were the only ones existing. Use the three rules already explained above.

You are now at liberty to redraw the whole circuit Fig.4 but substitute in place of V2, V3, R2, R3, R4 and R5 a different circuit section as in Fig.5 where the values of V1 and R1 were calculated as above. You will agree that Fig.5 already looks simpler than Fig.4.

Notice that we are not saying Fig.4 and Fig.5 are the same. No! they are very different circuits. What we are saying is that the currents and voltages in the right hand portion, outside that dotted ring, are the same in Figs.4 and 5.

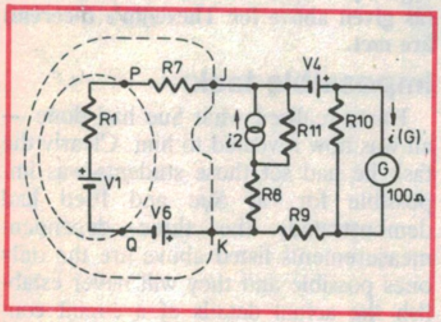

Fig.5: V1 and R1 are the Thevenin equivalent of the six components inside the dotted loop PQ in the circuit Fig.4. Next step is to choose points J and K to produce a still simpler equivalent to the real circuit.

But how does that help our quest? We wanted to know the current i(G)! Be patient — we'll get there.

Now select two other suitable points in Fig.5, say J and K. Through these we can draw a second dotted ring, following the same rules as before. By the same process as above, we now find suitable values for V1b and R1b such that one voltage source V1b with one resistor R1b in series is equivalent to everything inside the dotted loop JK (which includes the smaller loop PQ).

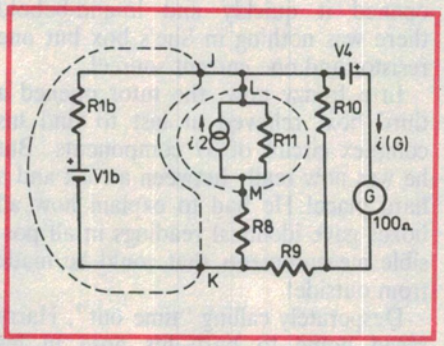

Fig.6: V1b and R1b are the Thevenin equivalent of everything within the dotted loop JK (including the loop PQ of Fig.5). The next step uses the dotted sub-loop LM.

The result is Fig.6, and Thevenin's theorem asserts that the current i(G) is the same in Figs.6, 5 and 4.

That current source

Now to do something about the current source i2. In Fig.6 we have redrawn slightly the connection to R11, carefully, without changing the circuit, so we can see points L and M and the little dotted sub-loop thru them.

Now, using Norton's theorem, (i.e., the external equivalence of Sue's circuit with the circuit produced by Fred), we can imagine substituting inside dotted loop LM one voltage source V1c in series with one resistor R1c. The values of these imaginary components are found by the three rules as before. The result is Fig.7, again without any change in the current i(G).

Clearly R1c and R8 are equivalent to a single resistor of numerical value (R1c + R8). By now it is clear how we should take the next step using points N and S, resolving (in our imagination) everything within that loop to one voltage source in series with one resistor.

Then the final step using points W and X, in like manner, would (in our imagination) reduce everything inside the dotted loop to one voltage source

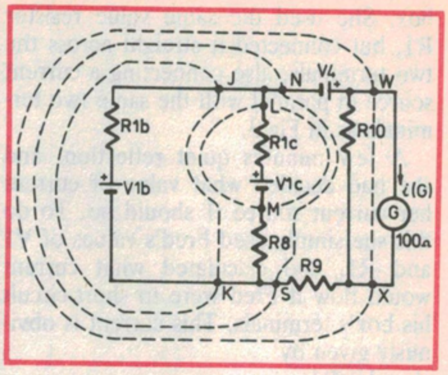

Fig.7: Substituting a voltage source V1c and series resistor R1c for the current source and parallel resistor of Fig.6, makes an equivalent circuit which has all the same type sources. Next we use dotted loop NS to reduce all within it to one voltage source and one series resistor. Lastly we use dotted loop WX to arrive at the final equivalent circuit.

V1e and one resistor R1e as in Fig.8. Again we can assert that the current i(G) in Fig.8 is the same as that in Fig.4.

Success!! Fig.8 immediately gives us the solution to the problem, for the current i(G) which we sought is simply:
i(G) = V1e / (R1e + 100 ohms)
that 100 ohms being the resistance of the instrument G.

We make no claim that the above use of Thevenin's and Norton's theorems is the best approach to the solution of every circuit problem. Indeed some questions are much better tackled by other methods. There is no universal "best way" to solve every problem.

But Thevenin-Norton is an excellent method and should be included in every student's repertoire (and aren't we all students?).

Applications

Some circuit problems lend themselves well to this approach, particularly those in which part of the circuit is to stay constant while another part is to take on a range of values, and results

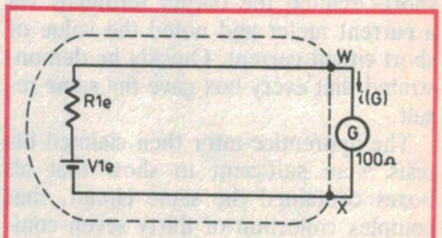

Fig.8: The final equivalent circuit, where current i(G) in the external circuit is the same as in Figs. 4,5,6, and 7.

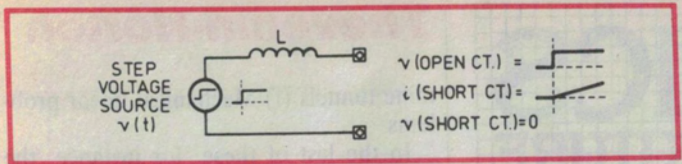

Fig.9: The Thevenin equivalent of a circuit with non-sinewave sources.

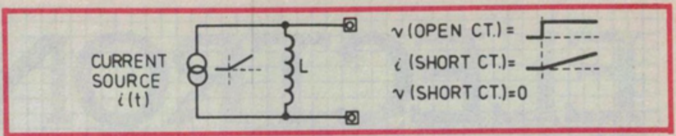

Fig.10: The Norton equivalent of Fig.9. Notice that the current generator i(t) in Fig.10 has a different waveform to the waveform of the voltage generator in Fig.9.

calculated for each case. For example if any one of a number of instruments "G", each having different resistance, were to be used in Fig.4, with no change in the rest of the circuit.

To calculate the current i(G) for each case, we could, as a mental exercise, carry out the imaginary reduction from Fig.4 to Fig.8. Having done that reduction once only, it would then be only a moment's work to find current i(G) for any number of resistance values of instruments "G".

Alternating currents

You may have a question: "What about AC voltage and current sources, capacitors and inductance? Can these be included?"

The answer is "Yes — definitely yes!" But with one condition: that each passive component be "linear", as defined above.

If in your problem all voltage and current sources are of sine waveform and the same frequency, f, (not necessarily in phase) then the concept of inductive reactance (XL=2πfL), capacitive reactance (XC = 1/2πfC), impedance (Z=SQRT[r2 + (XC — XL)2]) and the AC version of Ohm's Law (i=v/Z) may be used.

If on the other hand the voltage and/or current sources are mixed frequencies or of non-sine waveform the terms XL, XC and Z cannot be used (unless we use Fourier Transformation).

Different waveforms

However, no matter what the circumstance, we can always resort to the fundamental form of Ohm's law, viz:

vR = iR
vL = –L diL/dt
iC = C dvC/dt
vC = (1/C)Integral(iC)

In a circuit containing one or more inductive and/or capacitive elements, unless all sources are of sine waveform and the same frequency, then the equivalent Thevenin voltage source and the equivalent Norton current source will have different shape waveforms.

For example suppose the Thevenin voltage source is an upward step function and the series passive element is a pure inductance L, as in Fig.9. Then in the Norton equivalent, Fig.10, the same pure inductance L becomes the parallel passive element but the waveform of the current source becomes an uprising ramp function.

To see that this must be so, consider that the short-circuit current must be an uprising ramp function and the open circuit voltage has to be an upward step function in both Figs.9 and 10, and in the original circuit of which Figs.9 and 10 are equivalent reductions. Considering the presence of the inductance in the equivalent circuits, the terminal voltage and current demand the voltage waveforms shown.

Just what waveforms exist in the sources of the original un-reduced circuit depends on that circuit, but may well be different from that in either equivalent.

Generality

One last question can be heard faintly in the distance: "Is all this nice stuff restricted to electronic problems?" The answer is "No . . . . . . the theorems of Thevenin and Norton are perfectly general."

Indeed this is quite true. They have been applied successfully to solve such diverse puzzles as (a) Loudspeaker drive/suspension mechanics (b) Heat flow questions, as in the cooling of multiple transistors on heatsinks. (c) Mechanical rotational dynamics (d) Water flow question in irrigation systems (e) Air flow predictions in underground

What is a Current Source?

An ideal DC current source is an active electronic circuit with two output terminals, a nominal current rating io, and the property that no matter what value of load resistance is connected externally between those terminals, the current through the load is always exactly io.

You could regard the current source as a box which always puts out the same value of current.

This implies that the current source can automatically change its output voltage in such a manner that always the same current flows in the external load. This is not magic — such sources can be built, though the ones we build do not quite achieve the ideal characteristic of absolutely constant current io. Current sources have a very high (ideally infinite) value of output resistance.

Many different circuits are used to implement current sources, a simple example is shown here. In Fig.11, Q1 is a PNP transistor with high hFE, say 1000. Because of ZD1 there always exists 6 volts drop from A to B and assuming Q1 always has 0.6 volt drop from emitter to base, it follows that R2 always has 5.4 volts across it. But a constant voltage across a constant resistor R2 must mean a constant current through it. In this case the constant current thru R2, given by

i=V/R2
=(5.4 volts/5400 ohms)
=1.0 mA.

As the base current of Q1 is very small (only 1.0 microamp), it follows that the collector current, which is io, is always 1.0mA, at least within 0.1% — except when there is no external load connected, of course. Or a load of more than 15k, in fact.

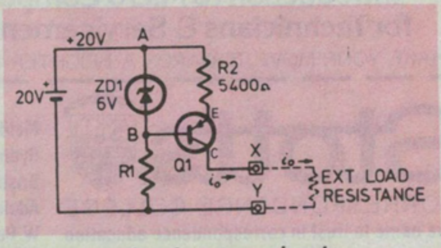

Fig.11: A simple approximate constant current source. Any resistance in the range zero ohms to about 15k placed across terminals XY will have (1.0 mA +/- 0.1%) flowing through it.

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mine tunnels (f) Mechanical linear problems

In the last of these, for instance, the version of Ohm's Law used is

fB = Bv

fm = m(dv/dt)

fk = k(integral(v))

where v = velocity; f = force; B = frictional damping; m = mass; k = spring constant

Here velocity and force are the analogues of electrical voltage and current respectively, while friction damping, mass and spring constant are respectively the analogues of electrical conductance, capacitance and "inverse inductance". The mechanical and analogous electrical systems are called "Duals".

Last words

Two final comments before we say "Enough!!" and resume a recumbent posture:

(1) Recall that we demand the restriction of "linearity" (implying constancy and reciprocity), on the components within our imaginative "black box", and in the Thevenin and/or Norton equivalent circuit, everything within those "dotted rings" in Figs.4 to 8.

However no such restrictions apply to external components and sources. Indeed the external load may be anything, active or passive, linear or non-linear. It may even contain non-linear resistors, diodes, transistors, vacuum tubes, generators or even regenerative electric motors. Anything.

(2) We have only said that the original circuit and the Thevenin and/or Norton equivalents are *equivalent at the terminals*. They are, in general not the same inside the "black box". For example the power dissipated within the "dotted ring" in the original real circuit may be very different to the power dissipated within the Thevenin equivalent.