impedance

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WHEN learning about electronics, one of the earliest things that we find is that for a resistor the voltage across it and the current driven through it are related by Ohm's Law

$$E = IR$$

where E is the voltage across a resistor with resistance R and I is the current. These three quantities are generally measured in volts, ohms and amps respectively. We then discover that capacitors and inductors produce a similar relationship, which is written as $E = IZ$ where Z is the impedance of the component and E and I are assumed to be sinusoidally varying a.c. voltage and current respectively.

The reason for this assumption is that non-sinusoidal waveforms are effectively a mixture of more than one frequency. Since the impedance of many components varies with frequency, the current flowing will not then be directly proportional to the driving voltage. It is worthwhile noting, as an aside, that the way that non-sinusoidal waveforms are dealt with is to break them up into the sum of sinusoidal parts, each of a different frequency. Each of these parts can then be dealt with using $E = IZ$, the current flowing can then be reconstructed by summing the currents of the individual parts.

In order to use the above formula we need to know the value of the impedance Z for the circuit. For a capacitor the impedance (often called the reactance) is given by $1/2\pi fC$, where f is the frequency of the signal in hertz, C is the capacitance of the capacitor in farads (a unit which we soon find out is about a million times larger than is useful), π is 3·1416. Similarly the impedance of an inductor is $2\pi fL$ where L is the inductance of the inductor in henrys.

Given this starting point it soon becomes apparent that there is much more to the impedance of capacitors and inductors than their simple numerical value. Consider, for example, the case when we have a capacitor and an inductor which both have an impedance of, say, 10Ω at a particular frequency that we shall apply. If we connect these components in series and apply an a.c. voltage we might expect that the impedance of the combination would be 20Ω, but this is not the case. The total impedance is in fact zero! Furthermore if we connect the components in parallel the impedance is not 5Ω, it is infinitely large.

The above example demonstrates that we require rather more information if we want to calculate the impedance of a combination of elements.

COMPLEX NUMBERS

We now introduce the concept of a complex number which consists of two parts: called the real part and the imaginary part. A complex number is written like this

$$x + yi$$

Here x is the real part and consists of a real number, whilst y is the imaginary part. i is the important symbol (j is often used instead) which represents a number which when squared gives minus one

$$i \times i = -1$$

If that is a little difficult to conceive of it doesn't matter—just think of it as a symbol which labels the imaginary part of the complex number. Examples of complex numbers are: $3 + 4i$, $10 - 3i$, $-3{\cdot}6 \times 10^4 + 6{\cdot}7i$. Real numbers such as -7 and 43·6 may also be thought of as complex numbers whose imaginary part is zero. Similarly there are numbers such as $6i$, $-0{\cdot}2i$ or even i (which is the same as 1i) in which the real part is zero.

ARITHMETIC

Fig. 1 demonstrates a way in which complex numbers can be shown on a diagram. In the figure the complex number is $3 + 4i$. It is represented by a line which goes from the origin of co-ordinates (marked 0) to the point which lies on the lines; real part = 3 and imaginary part = 4. This line has a certain length "r", and makes a certain angle to the real axis "θ". Note that the complex number can be specified in terms of r and θ, and these two numbers completely specify a particular complex number, just as x and y do.

Given the representation of a complex number as $x + yi$ or as an r and a θ, it is always possible to convert from one representation to the other.

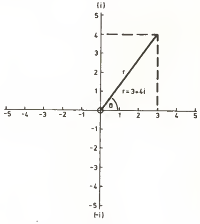


Fig. 1

Applying Pythagoras' Theorem to Fig. 1

$$r = \sqrt{x^2 + y^2}$$

This is by far the most often needed conversion. For the more mathematically minded we will give the other formulae

$$r\cos\theta = x$$
$$r\sin\theta = y$$
$$\tan\theta = y/x$$

Of these the last is the most important.

Addition of two complex numbers simply involves adding the real and the imaginary parts separately as shown below

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$

For subtraction you just subtract real and imaginary parts separately—be careful to get the signs right!

$$(x + yi) - (a + bi) = (x - a) + (y - b)i$$

Multiplication is a little more complicated

$$(x + yi) \cdot (a + bi) = (xa - yb) + (xb + ay)i$$

Unfortunately dividing complex numbers is more difficult than the preceding cases—hopefully the following steps should make the process clear. Assume we want to evaulate

$$\frac{x + yi}{a + bi}$$

First we multiply both the top and the bottom of this expression by a — bi. Since this is the same as multiplying the original expression by one, our division can now be written as

$$\frac{(x + yi) \cdot (a - bi)}{(a + bi) \cdot (a - bi)}$$

If we now multiply out $(a + bi) \cdot (a - bi)$ we get $a^2 + b^2$ which has no imaginary part at all, so our expression is the same as:

$$\frac{(x + yi) \cdot (a - bi)}{a^2 + b^2}$$

and we know how to multiply the top to get

$$\frac{(xa + yb) + (ya - xb)i}{a^2 + b^2}$$

and this is the same as

$$\frac{(xa + yb)}{a^2 + b^2} + \frac{(ya - bx)\, i}{a^2 + b^2}$$

There is just one more thing before we finish our maths lesson and that is how to multiply and divide complex numbers when they are in r and θ form. This is simpler than for numbers in x + yi form: to multiply you multiply the "r"s and add the "θ"s; to divide you divide the "r"s and subtract the "θ"s.

In these examples the complex numbers are written as (r,θ); thus $(2,36^\circ)$ stands for the complex number with $r = 2$ and $\theta = 36^\circ$.

(i) $(3,15^\circ) \cdot (4, -12^\circ) = (3 \cdot 4, 15^\circ - 12^\circ) = (12,3^\circ)$

(ii) $(16,186^\circ) \cdot (\tfrac{1}{2}, -26^\circ) = (16 \cdot \tfrac{1}{2}, 186^\circ - 26^\circ) = (8,160^\circ)$

(iii) $\dfrac{2,17^\circ}{5,27^\circ} = \dfrac{2}{5}, -10^\circ$

But remember that 360° is a full circle, so that -10° is the same as $+350^\circ$ so $\dfrac{2}{5}, -10^\circ = \dfrac{2}{5}, 350^\circ$

It is now time to use these numbers.

COMPLEX IMPEDANCES

Any impedance which is a combination of resistances, capacitances and inductances can be represented as one complex number. Sinusoidally oscillating voltages and currents are also represented by complex numbers. Using the arithmetic of complex numbers that we have described, it is now possible to use Ohm's law to give the correct answer and we can combine impedances in the same way as we used to combine resistances. Let's see how this works.

Resistors have no imaginary part to their impedance, it is just their resistance R.

Capacitors have no real part to their impedance, it is given by $-i/2\pi fC$. The symbols all have the same meanings as before.

Inductors too have no real part to their impedance, it is given by $2\pi fLi$.

It is easiest to represent voltages and currents in r and θ notation. First it is essential that you know about the phase difference between two sinusoidal waveforms of the same frequency.

The phase difference is given by the distance between the peaks of the two waveforms and is specified by an angle which is worked out by defining the angle between two successive peaks of the same wave to be 360°. Reference to Fig. 2 should make this clearer.

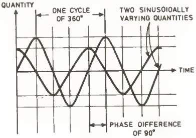


Fig. 2

To describe a voltage or current in terms of a complex number it is necessary to take one waveform in the circuit as a reference to which all the others will be referred. This reference value has no imaginary part, and its real part is just its peak value. All other voltages or currents are represented in r and θ notation by a complex number with r equal to the peak value and θ equal to the angle by which the waveform "leads" the reference waveform. By leads we mean that the angle is measured from a peak of the wave to the next peak in time of the reference wave. See Fig. 3 for an example of this.

With the set up we have just described, almost anything that you could have done with resistances and d.c. voltages can now be done for impedances and sinusoidal (i.e. one frequency) a.c. voltages.

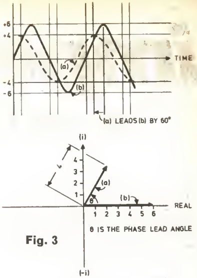


Fig. 3

TUNED CIRCUIT

The complex impedance of the series tuned circuit in Fig. 4 (a) is found simply by adding together the complex impedances of the capacitor and the inductor to get

$$\left(2\pi fL - \frac{1}{2\pi fC}\right) i$$

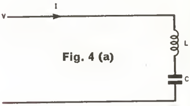


Fig. 4 (a)

Note that the impedance still has no real part. Hence in r and θ form it has

$$r = \left(2\pi fL - \frac{1}{2\pi fC}\right) \text{ and } \theta = 90^\circ.$$

You may notice that r might be negative in the above formula, in which case θ would be 270°—but negative r in the direction of 90° is the same as positive r in the direction of 270°. It isn't usually worth bothering about these things, they almost always work out alright in the end!

Now suppose that we want to know what current flows in the circuit. We know that $E = IZ$, so $I = E/Z$. Choose the input voltage to be the reference quantity for the circuit—it will then have $r = V$ (the peak value) and $\theta = 0^\circ$. To work out the current flowing we divide E by Z, remembering to divide the "r"s and subtract the "θ"s. So I has

$$r = \frac{V}{2\pi fL - \dfrac{1}{2\pi fC}} \text{ and } \theta = 0 - 90^\circ = -90^\circ.$$

So the current is 90° out of phase with the voltage, remembering that leading by —90° is the same as lagging by 90°. We have a rather peculiar expression for the peak value of the current (r). Notice how this expression is positive for high frequencies but negative for low ones. Thus the arrangement of the waveforms in

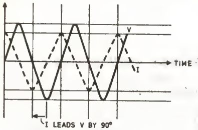


Fig. 4 (b)

Fig. 4 (b) is only valid for high frequencies. As f decreases, I suddenly becomes very large (when the bottom of the expression for r becomes zero) and then smaller again. However r is now negative so the phase changes by 180°—this is the same as saying that the waveform of the current becomes inverted.

In practical circuits of this nature there is always some resistance present so the change occurs gradually. Note that the impedance of this circuit goes to zero when

$$2\pi fL = \frac{1}{2\pi fC}$$

i.e. when

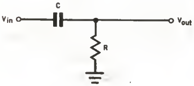
$$f = \frac{1}{2\pi\sqrt{LC}}$$

which is the well known resonant frequency.

SIMPLE R.C. CIRCUIT

The circuit shown in Fig. 5 is a very basic high pass filter. To find the current which flows we need to know the impedance of the combination, which is given by

$$Z = R + \left(\frac{-1}{2\pi fC}\,i\right) = R - \frac{1}{2\pi fC}\,i$$



Choosing V_{in} to be the reference quantity (which, you should remember means that $r = V_{in}$ and $\theta = 0°$) we can then say that the current flowing is given by

$$I = \frac{V}{Z} = \frac{V_{in}}{R - \frac{1}{2\pi fC}\,i}$$

Now the output voltage is produced by the current I flowing through the resistor R so, using Ohm's Law, we obtain

$$V_{out} = \frac{V_{in}R}{R - \frac{1}{2\pi fC}\,i}$$

To evaluate this we had better put it into r and θ form. $V_{in}R$ is simple—since it is just an ordinary number with no imaginary part it has $r = V_{in}R$ and $\theta = 0°$. To convert the bottom half of the expression we need to use the formulae for r and θ in terms of x and y that we mentioned earlier: namely $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$ (this last part can be done by scale diagram). Putting the x and y values of

$$R + \frac{1}{2\pi fC}\,i$$

into these formulae gives the value of r to be

$$r = R^2 + \frac{1}{4\pi^2 f^2 C^2}$$

and θ is going to be the angle for which

$$\frac{y}{x} = \frac{-1}{2\pi fRC}.$$

Using the rule for division in r and θ form we can now calculate the value of $\frac{V_{out}}{V_{in}}$ —the r part is

$$r = \frac{R}{\sqrt{R^2 + \frac{1}{4\pi^2 f^2 C^2}}}$$

and the θ part is minus the angle for which

$$\frac{y}{x} = \frac{-1}{2\pi fRC}$$

—if you draw a diagram you can see that this is the same as the angle for which

$$\frac{y}{x} = \frac{1}{2\pi fRC}$$

The r value gives us the amount by which the amplitude of the voltage is decreased. When

$$R^2 = \frac{1}{4\pi^2 f^2 C^2}$$

this attenuation factor is about ·707 (or, using the decibel scale, about —3dB). Rearranging this formula and getting rid of all the squares gives

$$f = \frac{1}{2\pi RC}$$

This is often called the break point for the filter.

What do all these complicated-looking formulae mean as regards the performance? Well, when the frequency is very high the

$$\frac{1}{4\pi^2 f^2 C^2}$$

term is very small and so $\frac{V_{out}}{V_{in}}$ becomes very close to one. This indicates that high frequencies pass through the filter almost unobstructed. In contrast, when f is very small the

$$\frac{1}{4\pi^2 f^2 C^2}$$

term is going to be far larger than the R^2 term so we can ignore the R^2 term without too much loss of accuracy. We then have

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{\frac{1}{4\pi^2 f^2 C^2}}} = 2\pi fRC$$

Notice that the $\frac{V_{out}}{V_{in}}$ figure halves every time the frequency halves. This sort of relationship is best shown on a decibels versus logarithmic frequency plot as shown in Fig. 6 (a).

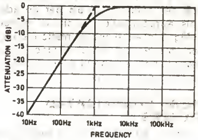


Fig. 6 (a)

Taking 0dB at the input level, the output level of the filter is fairly constant down to just above the break point: the output then curves down, finally falling off at about 6dB per octave (halving of frequency) which is 20dB per decade.

We have not yet used the information we have calculated about θ. For actual values of R,C and f, θ can be evaluated either by drawing a diagram or by working out

$$\theta = \arctan\frac{1}{2\pi fRC}$$

on a scientific calculator. We can see roughly what is going to happen: at very high frequencies when the filter is passing almost all of the input voltage, y/x is very small which means that the output voltage has almost the same phase as the input voltage. As the frequency decreases it will reach the break point where

$$\frac{1}{2\pi fRC} = 1$$

—this means that the output will lead the input by 45°. As the frequency keeps on decreasing the phase lead will continue increasing, getting ever nearer to 90° but never quite getting there as shown in Fig. 6 (b).

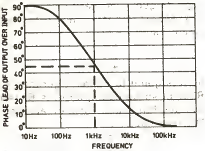


Fig. 6 (b)

When working with filters such as this one, it is generally true that the attenuation versus frequency graphs (on logarithmic scales) can be simplified considerably. To do this you just assume that the response is flat down to the break point, whereupon it falls off immediately at a rate of 6dB per octave = 20 dB per decade. This approximation is shown dotted in Fig. 6 (a)—the approximation is very accurate except for a decade or so around the break point when it can be up to 30 per cent out.

MORE COMPLICATED FILTERS

As a slightly more complicated example let us try to design a filter which passes high frequencies unattenuated and attenuates low frequencies by 10. We want the middle point to be at 1kHz (the full attenuation is 20dB so call the mid point the 10dB attenuation point). We would also like some idea of the phase performance.

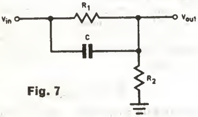


Fig. 7

The obvious way to do this is shown in Fig. 7. At very low frequencies the effect of the capacitor is insignificant so

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2},$$

which we want to equal

$$\frac{1}{10}.$$

Choose as fairly sensible values $R_1 = 9k\Omega$, $R_2 = 1k\Omega$. Whether these are sensible will, of course, depend on the impedance of the source we are using to drive the filter and the impedance that is being driven by the filter. Say for the sake of simplicity that the driving impedance is a few ohms and that the driven impedance is at least several tens of kilohms. Now down to work:

Using the symbol $R_1 \| C$ to mean the impedance of the parallel combination of R_1 and C we have

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_2 + R_1 \| C}$$

But $R_2 = 1k\Omega$ and $R_1 = 9k\Omega$ so

$$R_1 \| C = \frac{9000\left(\frac{-1}{2\pi fC}\right) i}{9000 - \frac{1}{2\pi fC} i}$$

and

$$\frac{V_{out}}{V_{in}} = \frac{90000 - \frac{5}{\pi fC} i}{90000 - \frac{50}{\pi fC} i}$$

The "r part" of this expression can be found by dividing the r part of the top by the r part of the bottom

$$r = \frac{\sqrt{(90000)^2 + \left(\frac{5}{\pi fC}\right)^2}}{\sqrt{90000^2 + \left(\frac{50}{\pi fC}\right)^2}} = \frac{1}{\sqrt{10}}$$

since we require this to be equivalent to an attenuation of 10dB when f = 1kHz and 10dB is a voltage ratio of $\sqrt{10}$: 1.

Square both sides and multiply out which gives

$$9 \times 81 \times 10^8 = \frac{2500 - 250}{\pi^2 f^2 C^2}$$

this is for f = 1kHz so

$$C^2 = \frac{2250}{9 \times 81 \times 10^8 \times \pi^2 \times 10^2}$$

$$= 3{\cdot}127 \times 10^{-18}$$

and finally we get to the value of the capacitance $C = 5{\cdot}6 \times 10^{-8}$ farads = 0·056μF. If we substitute this value into the original formula for the r part we get

$$\frac{\sqrt{(90000)^2 + \frac{8{\cdot}1 + 10^{14}}{f^2}}}{\sqrt{(90000)^2 + \frac{8{\cdot}1 \times 10^{12}}{f^2}}} = \frac{\sqrt{1 + \frac{10^8}{f^2}}}{\sqrt{1 + \frac{10^7}{f^2}}}$$

A graph of this is the attenuation of the filter, as shown in Fig. 8 (a).

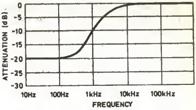


Fig. 8 (a)

PHASE PERFORMANCE

To get some idea of how the phase difference between the input and the output varies with frequency we see that at high frequencies the capacitor is going to have far more effect than R_1 and so the phase shift will go to zero, just as it did for the simple high-pass filter. Also at very low frequencies the effect of the capacitor will be negligible and the phase shift will go to zero again. What happens in between? If we substitute the value for the capacitance into

$$\frac{V_{out}}{V_{in}} = \frac{90000 - \frac{5}{\pi fC} i}{90000 - \frac{50}{\pi fC} i}$$

we end up with

$$\frac{V_{out}}{V_{in}} = \frac{1 - \frac{316}{f} i}{1 - \frac{3160}{f} i}$$

From this we can work out the phase shift for any frequency: for example at 1kHz we have

$$\frac{V_{out}}{V_{in}} = \frac{1 - 0{\cdot}316\ i}{1 - 3{\cdot}16\ i}$$

Now θ for 1 — 0·316i is about — 17½° and θ for 1 — 3·16i is about — 72½°, so θ for

$$\frac{V_{out}}{V_{in}} = -17\tfrac{1}{2}° - (-72\tfrac{1}{2}°) = 55°$$

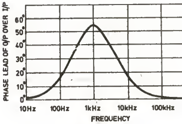


Fig. 8 (b)

In fact this is the maximum phase shift for any frequency. A graph of phase shift versus frequency is also shown on Fig. 8 (b).