...al Audio

...FT: Part I

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...ing I sat down to write this ar... on the FFT, which stands for Fast Fourier Transform. The subject matter is important and relevant but it requires the use of mathematics in order to explain it elegantly.

The key element in much of the mathematics of signal processing and filter design is the use of the concept of the square root of -1 ($\sqrt{-1}$). If you have not seen the idea before, you will probably (and reasonably) be befuddled. There is no number which when multiplied by itself will give a negative number. A negative multiplied by a negative is a positive number. How can we find a number that has a square which is negative? The answer is that we create such a number. Engineers call it operator "j" and mathematicians call it "i." This is a created concept for which there is no proof. We simply say "there exists this thing, which we create, that has the property of $j \times j = -1$. You cannot find it in the laboratory or the grocery store.

We say that j is one unit of "imaginariness." The number 7j is seven units of "imaginariness." Ordinary numbers are now given the property of "realness" (the adjective form is dropped so that we have imaginary numbers and real numbers). By analogy, we could define location in space as real miles and imaginary miles. This concept would be useful if we assign real miles to be along the East-West axis and imaginary miles to be along the North-South axis. This allows us to say that Boston is approximately +150 +50j miles from New York City. Notice that the composite number is a vector since it contains both distance and direction. It is a two-dimensional representation. It is equivalent to saying that Boston is about 200 miles northeast by east from New York City. The first description is the "rectangular" form; the second is the polar form. Both the polar and rectangular forms are vectors because they contain information about size and direction. Ordinary numbers are one-dimensional in that they only give the amount of something.

As we go forward in the discussion, you can think of real numbers as "red" stuff and imaginary numbers as "blue" stuff. Terms containing both real and imaginary numbers are considered "complex." For addition, we add the amount of red stuff from one number to the amount of red stuff from the other number to give the total amount of red stuff. The same is true for the blue stuff. This is illustrated below:

$$\begin{array}{r} 3 + 8j \\ -1 + 1j \\ \hline 2 + 9j \end{array}$$

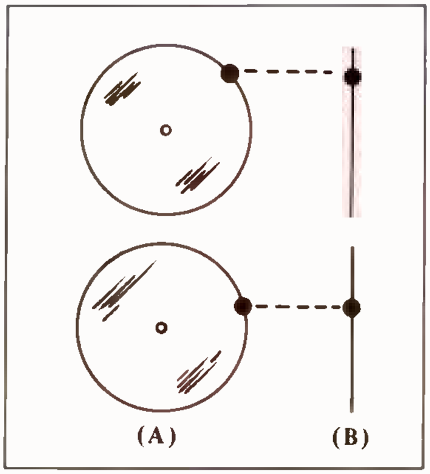

Figure 1. Rotating glass disc; a) front view, b) edge view.

Simple, is it not? Multiplication is a little more complicated since we must do the multiplication with all of the cross terms. Consider the following example:

$$(3 + 8j) \times (-1 + 1j)$$

Each of the terms from the first expression must be multiplied by each of the terms from the second expression. This gives the following:

$$(3 \times -1) + (8j \times -1) + (3 \times 1j) + (8j \times 1j)$$

The first term is 3 real times -1 real which is -3 real. The second term is 8 imaginary times -1 real which is -8 imaginary. The next term is 3 imaginary; and the last term is 8 imaginary squared. But, the imaginary times the imaginary is -1; thus the last term is -8 real.

We are now in a position to manipulate complex numbers. You may ask why we have bothered to introduce the two-dimensional aspects of numbers. The answer is: the sine wave.

COMPLEX SINE WAVE

The sine wave is a complicated function which changes its value according to some property that is difficult to understand. One way of simplifying the sine wave is to consider the following idea.

Take a transparent glass disc and paint a black dot on the rim. Now turn the disc edgewise in the vertical position so that you see the dot looking through the glass. The glass looks like a line when viewed from the edge. Let us rotate the glass disc at a constant velocity. What does the dot do? It goes up and down. Figure 1 shows the two views of the disc at different instants of time. The left part of the figure is the frontal view, the right part the edge view. As we watch the dot move up and down in the edge view we notice that it is a sine wave in time! In other words, it is a sine wave rotation projected onto a single dimension.

If we were to look at the edge of the disc from the top instead of the side, we would see the same thing. However, the phase would be delayed by 90 degrees, so instead of a sine wave, we would see a cosine wave. By looking from both the top and the side, we can determine the exact location of the dot; in contrast, by looking only from one direction we cannot tell its exact location. There are two locations for each projection value. When the dot is in the center of the up-down range, the actual location of the dot might be at 0 degrees or at 180 degrees. The front view allows us to resolve the difference, since it will be either fully left or fully right. The two views of the disc are like the "red" stuff and "blue" stuff. Or, we can assign one projection to be real numbers and the other projection to be imaginary numbers.

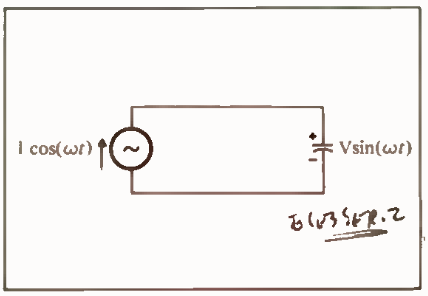

Figure 2. A simple circuit.

Looking at the disc from the frontal view, when the dot is at 45 degrees we could say that it is located at 0.707 +0.707j. When it is at 90 degrees we would say that it is at 0 + 1j; at 135 degrees it is at -0.707 + 0.707j; at 180 degrees it is at -1 + 0j. From now on, we will consider all sine waves as coming from the rotating disc. To get back to the real world, we say that the signal is either a left-right projection or an up-down projection of a two-dimensional process. The angle of the dot defines the final result. Notice that the size of the disc becomes the magnitude of the number.

Mathematically, we say that the "signal" is defined as the following:

$M \cos(\theta) + j M \sin(\theta)$

where:

M = radius of the disc and θ is the angle of the dot.

For any given radius and any given angle, we know the location of the dot. At first glance, this would appear to be a lot of trouble for us engineers. The trouble is a real blessing, however, because it makes many mathematical tasks very simple. I admit that these ideas can be more than a little mysterious if you have not seen them before.

I was motivated to teach this set of ideas because they are applicable to much more than the FFT. They are used in digital filters, ordinary analog filters, signal processing, and many other aspects of audio engineering. Rather than rush to the subject of the FFT, which will be discussed next month, let us stop and have a little practice with these ideas in the context of simple analog circuits.

COMPLEX VOLTAGES AND CURRENTS

Let us assume a set of circuits that are excited only by sine waves of the form I sin (ωt), I cos (ωt), V sin (ωt) and V cos (ωt). Instead of thinking of these as sine waves, we could think of them as coming from our rotating disc.

To convert from one projection to another, we need to convert from "red" stuff to "blue" stuff; or we need to convert from real numbers to imaginary numbers (or vice versa). Notice that multiplying by j does this conversion. A real number multiplied by a complex number becomes complex; a complex number multiplied by a complex number becomes real.

The act of phase shifting is the same as multiplication by a complex number. A full 90 degree shift requires a multiplication by j but a 45 degree shift requires a multiplication by (0.707 + 0.707j). A capacitor having an impedance of 10 ohms at a given frequency is thus represented as having a complex impedance given by:

$Z = -10j.$

A 1 amp current source is represented by:

$I = 1.$

And the resulting voltage by ohms law becomes:

$V = Z I = -10\,j$ volts.

The magnitude 10 gives us the size of the sine wave and the -j gives us the phase.

The normal problem with RLC circuits and sine waves is that we cannot add two voltages or currents in terms of magnitudes because there is the phase shift to consider. We can, however, add the sine part to other sine parts and the cosine part to other cosine parts. Complex numbers are therefore perfect for representing both parts in one number; a single number contains both the magnitude and the phase. This is the real power of these numbers.

We can further illustrate the power by taking the simple lowpass filter of FIGURE 3. This would be a simple voltage divider if both components had been resistors. It is still a voltage divider if we use the complex impedance for the components. The gain or attenuation is represented as:

$$G\,(\text{complex}) = \frac{Z_2}{Z_1 + Z_2} = \frac{V_o}{V_i}$$

where:

the impedance of the resistor, Z_1, is defined as $Z_1 = R$ and the impedance of the capacitor, Z_2, is defined as:

$$Z_2 = \frac{1}{j\,\omega C}$$

where:

j is our complex number, ω is $2\pi \times$ frequency in radians, and C is the capacitance in farads.

This results in the expression:

$$G = \frac{\left(\frac{1}{j\omega C}\right)}{\left(\frac{1}{j\omega C}\right) + R} = \frac{1}{1 + j\omega RC}$$

We got this result by substitution and then by multiplying the numerator and denominator by $j\omega C$. The final result tells us all there is to know about the circuit. When ω is very low,

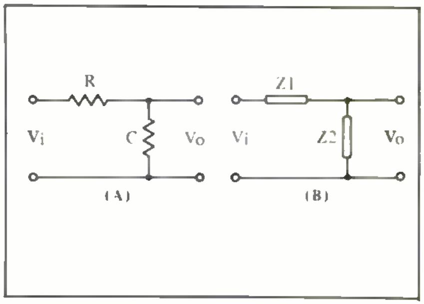

Figure 3. A low pass filter.

the gain is approximately 1; when ω is very large, the gain is approximately $1/(j\omega RC)$. There is a 90 degree phase shift, and each doubling of frequency results in a halving of gain. When ω = RC, the gain G becomes 0.500 - 0.500j which has a phase shift of 45 degrees and a gain of 0.707 (magnitude = -3 dB).

Our little example is rather trivial but it does demonstrate the incredible power of having a single number to contain both magnitude and phase. We can multiply two complex gains to get a net gain which is also complex. We can do circuit analysis as if each component were like a resistor but we use the complex impedance. This makes capacitors and inductors very simple because the phase is included in the impedance.

In digital signal processing, the element of delay is like phase shift. A complex digital filter, which is made up of delays, can be analyzed by changing each delay to an equivalent phase shift. A delay of 10 degrees corresponds to a complex number of 0.984 - 0.174j. Therefore, multiplying this gain by the input signal will result in an output signal which has the same magnitude but a delay of 10 degrees.

If you have not had the mathematical training in this area it would be hard to completely understand the ideas. My main hope is that you would at least believe that a complex number is not just some crazy idea of a mathematician to make an engineer's life harder. If the word complex makes you nervous because you do not believe in the concept of the $\sqrt{-1}$, then just change the names real and imaginary to red and blue stuff or sine stuff and cosine stuff.

Sometimes, simple ideas are hard to understand only because the words are strange. If that is your problem, just change the words. Next month we will continue this discussion but with an application to the Fourier transform and Fast Fourier transform. ■