Designer's guide to **advanced vibration analysis**, Part II

Last month's column discussed the basics of simulating resonant or natural vibration frequencies as an introduction to performing advanced vibration analysis. Recall that advanced vibration-analysis problems can be solved with what are called the *normal modes* and the *Mile's Equation* approaches.

In detail, the normal-modes method suits large, complex, multidegree of freedom systems (>100 DOFs) that are typically analyzed using FEA techniques. This approach introduces a transformation of coordinate approximations to decouple the differential equations of motion and make the problem easier to solve.

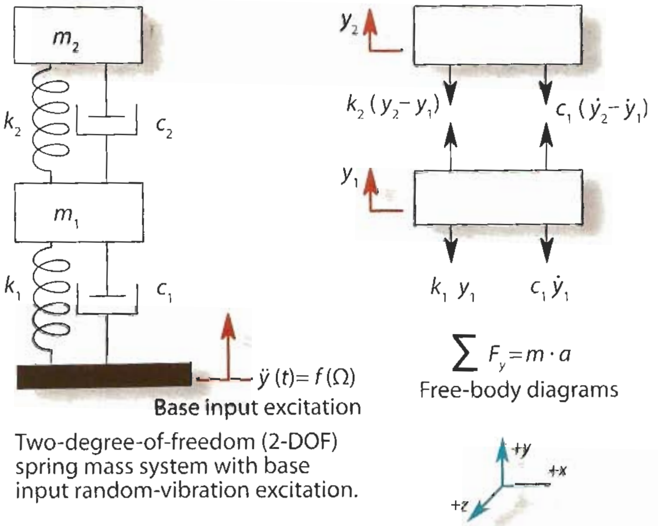
The Mile's Equation approach, on the other hand, involves a simplified linear-static approximation for FEA models of large complicated systems that contain an excessively large number of DOFs, which makes an explicit computer solution difficult to obtain. The approach is based on statistical analyses of induced acceleration spectra with a three-sigma distribution. The software approximates an equivalent *g* loading using the power-spectral density (PSD) criteria at the resonant frequency in each orthogonal direction of interest. This equivalent *g* load is sometimes referred to as the random-vibration load factor (RVLF).

Following is a summary of the use of these approaches to approximate the response caused by a random vibration input on the simple 2-DOF spring-mass system.

In the normal-modes approach,

David R. Dearth
President
Applied Analysis & Technology
Huntington Beach, Calif.
AppliedAT@aol.com or www.AppliedAnalysisAndTech.com

Edited by **Leslie Gordon**



Two-degree-of-freedom (2-DOF) spring mass system with base input random-vibration excitation.

In this simple 2-DOF spring-mass system, m_1 and m_2 represent the individual concentrated mass — units are lb-sec²/in.; k_1 and k_2 represent the spring rate or stiffness — units are lb/in.; c_1 and c_2 represent the damping — units are lb-sec/in.; and $y_1(t)$ and $y_2(t)$ represent the displacement of each mass, m_1 and m_2, as a function of time — units are in inches.

summing the forces in the vertical direction gives these equations of motion, which represent the free vibration, natural frequencies of the undamped system:

$$\sum F_y = m \times a$$

$$-k_1 y_1 + k_2(y_2 - y_1) = m_1 \ddot{y}_1$$
$$-k_2(y_2 - y_1) = m_2 \ddot{y}_2$$

$$\text{In matrix format} \Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & +k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$$

For this sample problem, $m_1 = m_2 = 90$ lb-in./sec²; $k_1 = 125{,}000$ lb/in.; and $k_2 = 200{,}000$ lb/in.

Also, viscous damping = $\beta = c/c_0 = 0.012$; the PSD input excitation $f(\Omega) = 0.20\ g^2/\text{Hz}$.

Next comes solving for the natural frequencies (eigenvalues) and associated mode shapes (eigenvectors).

The column matrix of natural frequencies, $\{\omega_n\}$, is:

$$\{\omega_n\} = \begin{pmatrix} 24.2583 \\ 72.4215 \end{pmatrix} \text{rad/sec}$$

And normalizing the eigenvectors matrix, $[\emptyset_r]$ brings:

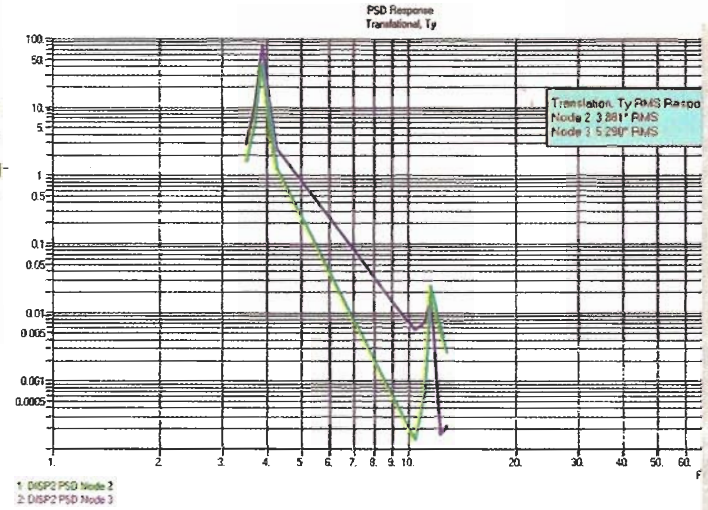
$$[\phi_r] = \begin{bmatrix} 1.0 & 1.0 \\ 1.3602 & -0.7352 \end{bmatrix}$$

Or, in terms of cycles/sec:

$$f_n = \frac{\omega_n}{2\pi}, \text{ cycles/sec (a.k.a. Hz);}$$

$$\{f_n\} = \begin{pmatrix} 3.861 \\ 11.526 \end{pmatrix} \text{Hz}$$

These are the results for the PSD response "y" deflections after the 2-DOF spring-mass system has been processed in Nastran using dynamic-response analyses.



Comparing hand results to FEA

	FEA % difference	Normal-modes hand-estimate rms	Mile's Equation hand-estimate rms	FEA model inch rms (Relative)
Mass 1:		3.869 in. −0.31%	3.956 in. +1.93%	3.881 in. n/a
Mass 2:		5.259 in −0.059%	5.192 in. -1.86%	5.290 in. n/a

The table shows the percent differences between hand estimates and the results of an FEA model of the 2-DOF spring-mass system processed in Nastran.

Without going through all the matrix arithmetic:

The modal-participation factors $\{\Gamma_r\}$ is:

$$\{\Gamma_r\}=[\phi_r]^T\{m\}=\begin{pmatrix}212.42\\23.83\end{pmatrix}$$

And the generalized mass matrix, $[M_r]$, is:

$$[M_r]=[\phi_r]^T[m][\phi_r]=\begin{bmatrix}256.51 & 0\\0 & 138.65\end{bmatrix}$$

Next comes approximating the mean-squared response of the 2-DOF system with this summation:

$$\bar{y}_i^2=\frac{f(\Omega)}{8\beta}\sum_{r=1}^{2}\frac{\phi_{ir}^2}{\omega_r^3}\frac{\Gamma_r^2}{M_r^2}$$

$$\text{For mass M1: } \bar{y}_1^2=14.968\text{ in.}^2 \xrightarrow{\text{yields}} y_1=\sqrt{14.968}=3.869\text{-in. rms}$$

$$\text{For mass M2: } \bar{y}_2^2=27.66\text{ in.}^2 \xrightarrow{\text{yields}} y_2=\sqrt{27.66}=5.259\text{-in. rms}$$

In contrast, use the Mile's Equation to compute the g_{rms} value, sometimes called a random-vibration load factor (RVLF), based on the fundamental resonant frequency of a single-degree-of-freedom spring-mass system. The Mile's Equation is:

$$g_{rms}=\sqrt{\frac{\pi}{2}f_n Q\Omega}$$

where f_n = fundamental natural frequency = 3.861 Hz; Q = transmissibility (amplification) at resonance = $1/(2\times\beta) = 1/(2\times0.012) = 41.67$; and Ω = input power spectral density (PSD) = 0.20 g^2/Hz.

Substituting gives $g_{rms} = 7.109$.

Next, estimate the static deflections at each mass:

$$\underline{\text{For Mass 1}}:\ y_1=\frac{(w_1+w_2)}{K_1}g_{rms}=\frac{69{,}522\text{ lb}}{125{,}000\text{ lb/in.}}g_{rms}=3.956\text{-in. rms}$$

$$\underline{\text{For Mass 2}}:\ y_2=y_1+\frac{w_2}{K_2}g_{rms}=y_1+\frac{34{,}776\text{ lb}}{200{,}000\text{ lb/in.}}g_{rms}=5.192\text{-in. rms}$$

For a sanity check, an FEA approach would entail developing an FE idealization of the 2-DOF spring-mass system to approximate the static deflections and resonant-vibration frequencies, processing the dynamic response due to random vibration, and comparing results. **MD**

Random-vibration analysis and overall rms values

- In the FEA model of the 2-DOF spring-mass system, we selected a range of frequencies limited to ±10% on either side of the two resonant frequencies so the FEA results more closely matched the arithmetic from the normal-modes approach. To more accurately represent random-vibration tests of an actual structural model, it is necessary to include the response between fundamental resonances in the root-mean-squared (rms) average.
- Because using structural damping is difficult to estimate for dynamic-response problems and might lead to inaccurate results, it is better to use viscous damping.
- Actual environmental tests of structures might be limited by the capacity of the shaker table.
- To account for the probability of peak acceleration levels, it is typical to multiply the rms response acceleration levels by three to reach three-sigma probability on deflections and stress levels.

DOWNLOAD FILES TO TRY AN ANALYSIS

Download the run notes and the 2-DOF spring-mass FEA model, "Random_2DOF_v2010.mod," by contacting *AppliedAT@aol.com*. User notes and input files for Patran MD-Nastran 2008r3 are also available. Patran users can download neutral run files "Resonant_Modes_2DOF.bdf" and "Random_2DOF_DiscretFreq.bdf". The input data files come in Nastran format and are small enough for the free demo version of Femap v10, available from *http://tinyurl.com/2cgcyts*.