A Bessel Filter Crossover, and Its Relation to Other Types[1]

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One of the ways that a crossover may be constructed from a Bessel low-pass filter employs the standard low-pass to high-pass transformation. Various frequency normalizations can be chosen for best magnitude and polar response, although the linear phase approximation in the passband of the low-pass is not maintained at higher frequencies. The resulting crossover is compared to the Butterworth and Linkwitz-Riley types in terms of the magnitude, phase, and time domain responses.

A REVIEW OF CROSSOVERS

There are many choices for crossovers today, due especially to the flexibility of digital signal processing. We now have added incentive to examine unconventional crossover types. Each type has its own tradeoffs between constraints of flatness, cutoff slope, polar response, and phase response. See [1] and [2] for more complete coverage of crossover constraints and types. Much of the content of this paper is closely related to previous work by Lipshitz and Vanderkooy in [3].

Our sensitivity to frequency response flatness makes this one of the highest priorities. It is often used as a starting point when choosing a crossover type.

Cutoff slopes of at least 12 dB per octave are usually chosen because of limitations in the frequency range that drivers can faithfully reproduce. Even this is less than optimal for most drivers.

Polar response is the combined magnitude versus listening angle from noncoincident drivers [4]. The ideal case is a large lobe in the polar response directly in front of the drivers, and happens when low-pass and high-pass outputs are in-phase.

The phase response of a crossover is one of its most subtle aspects, and so is often ignored. A purely linear phase shift, which is equivalent to a time delay, is otherwise inaudible, as is a small non-linear phase shift. Still, there is evidence that phase coloration is audible in certain circumstances [5], and certainly some people are more sensitive to it than others.

A first-order crossover is unique, in that it sums with a flat magnitude response and zero resultant phase shift, although the low-pass and high-pass outputs are in phase quadrature (90 degrees), and the drivers must perform over a huge frequency range. The phase quadrature that is characteristic of odd-order crossovers results in a moderate shift in the polar response lobe.

In spite of this, third-order Butterworth has been popular for its flat sound pressure *and* power responses, and 18 dB per octave cutoff slope.

Second-order crossovers have historically been chosen for their simplicity, and a usable 12 dB per octave cutoff.

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Fourth-order Linkwitz-Riley presents an attractive option, with flat summed response, 24 dB per octave cutoff, and outputs which are always in phase with each other, producing optimal polar response.

Steeper cutoff slopes are known to require higher orders with greater phase shift, which for the linear phase case is equivalent to more time delay.

A number of other novel and useful designs exist which should be considered when choosing a crossover. Generating the high-pass output by subtracting the low-pass output from an appropriately time-delayed version of the input results in a linear phase crossover, with tradeoffs in cutoff slope, polar response, and flatness [1]. Overlapping the design frequencies and equalizing the response can result in a linear phase crossover [3], with a tradeoff in polar response. A crossover with perfect polar response can be designed with a compromise in phase response or cutoff slope [6].

BESSEL LOW-PASS AND HIGH-PASS FILTERS

The focus of this paper is on crossovers derived using more traditional methods, which begin with an all-pole lowpass filter with transfer function of the form $\frac{1}{p(s)}$, where $p(s)$ is a polynomial whose roots are the poles.

The Bessel filter uses a $p(s)$, which is a Bessel polynomial, but is more properly called a Thomson filter, after one of its developers [7]. Still less known is the fact that it was reported several years earlier by Kiyasu [8].

Bessel low-pass filters have maximally flat group delay about 0 Hz [9], so the phase response is approximately linear in the passband, while at higher frequencies the linearity degrades, and the group delay drops to zero (see Fig. 1 and 2). This nonlinearity has minimal impact because it occurs primarily when the output level is low. In fact, Bessel low-pass and all-pass filters may be used purely for audio time delay, as described in [10].

The high-pass output transfer function may be generated in different ways, one of which is to replace every instance of s with $\frac{1}{s}$. This popular method results in the general transfer function (1); (2) is a fourth-order Bessel example.

$$\frac{1}{c_0 + c_1\cdot\left(\frac{1}{s}\right) + c_2\cdot\left(\frac{1}{s}\right)^2 + \dots + c_n\cdot\left(\frac{1}{s}\right)^n} = \frac{s^n}{c_n + c_{n-1}\cdot s + c_{n-2}\cdot s^2 + \dots + c_0\cdot s^n} \qquad (1)$$

$$\frac{1}{1 + \left(\frac{1}{s}\right) + \frac{9}{21}\cdot\left(\frac{1}{s}\right)^2 + \frac{2}{21}\cdot\left(\frac{1}{s}\right)^3 + \frac{1}{105}\cdot\left(\frac{1}{s}\right)^4} = \frac{s^4}{\frac{1}{105} + \frac{2}{21}\cdot s + \frac{9}{21}\cdot s^2 + s^3 + s^4} \qquad (2)$$

Note the reversed coefficient order of the high-pass, and an nth-order zero at the origin. Characteristics of the low-pass with respect to 0 Hz are preserved somewhat, but in the high-pass they are with respect to infinite frequency instead. The nature of the response of the high-pass follows from equation (3) below.

$$p\left(\frac{1}{j\omega}\right) = p\left(-j\omega_h\right) , \quad \omega_h = \frac{1}{\omega} \tag{3}$$

The magnitude responses are mirror images of each other on a log-frequency scale; the negative sign has no effect on this. The phase of the low-pass typically *drops* near ω_o from an asymptote of zero as the frequency is *increased*, and asymptotically approaches a negative value. However, the phase of the high-pass at ω_h is the negative of the low-pass at ω, which follows from the negative sign in (3). It *rises* from zero at high frequency, and approaches a positive asymptote as the frequency is *decreased*. This results in offset curves with similar shape. Any asymmetry of the s-shaped phase curve is mirrored between the low-pass and high-pass. See Fig. 5 for a second-order example, where the phase curve has inherent symmetry.

One special case is where the denominator polynomial p(s) has symmetric coefficients, where the n^{th} coefficient is equal to the constant term; the $(n-1)^{st}$ coefficient is equal to the linear term, etc. This is the case for Butterworth and therefore the Linkwitz-Riley types [3]. A fourth-order Linkwitz-Riley is given as an example in equation (4).

$$\frac{1}{1 + 2\cdot\sqrt{2}\cdot s + 4\cdot s^2 + 2\cdot\sqrt{2}\cdot s^3 + s^4} \tag{4}$$

When this is the case, coefficient reversal has no effect on p(s), and the high-pass differs from the low-pass only in the numerator term s^n . This numerator can easily be shown to produce a constant phase shift of 90, 180, 270, or 360 degrees (which is in-phase for a constant frequency), with respect to the low-pass, when frequency response is evaluated for $s = j\omega$ ($\omega = 2\pi f$). For the second-order case $(j\omega)^2 = -\omega^2$ and the negative sign indicates a polarity reversal (or 180-degree phase shift at all frequencies).

NORMALIZATIONS

Transfer functions may be frequency normalized by replacing all instances of s with s/ω_o for some $\omega_o > 0$. This is equivalent to shifting the response curve right ($\omega_o > 1$) or left ($\omega_o < 1$) when viewed on a log-frequency scale, and so changes the overlap between low-pass and high-pass. Normalizations may be based on the magnitude response of either output at ω_o , flattest summed response, or any other criteria.

Once the function is normalized, renormalizing with ω_o has the effect of centering the response about ω_o , so that it is actually denormalized.

In this paper the low-pass will be normalized with s/ω_o , while the high-pass will be normalized by replacing s with $s\omega_o$, and the overlap is then proportional to ω_o^2 . Transfer functions and graphs have a design frequency of $\omega_o = 1$ (f = 1/2π Hz), although other values of ω_o are used temporarily in the normalization process.

Normalization does not change the shape of the low-pass or high-pass magnitude or phase response, only their design frequencies and summed response. To see this, note that when we normalize a transfer function, we get g(s)= f(s/ω_o). If we now consider the response *about* ω_o , and about $\omega_f = 1$, we may substitute $j\omega_f\omega_o$ for s in g(s). Then we get:

$$g(j\omega_f \cdot \omega_o) = f\left(\frac{j\omega_f \cdot \omega_o}{\omega_o}\right) = f(j\omega_f) \qquad (5)$$

The magnitude and phase responses about ω_o are also about $\omega_f = 1$, and are identical to the original response about $\omega = 1$. Group delay, being the negative derivative of phase with respect to ω, will also be scaled in magnitude, but otherwise retains its shape.

Normalization influences the symmetry of p(s), but perfect symmetry is not achievable in general. This means that it will not always be possible to make the low-pass and high-pass phase response differ *exactly* by a constant multiple of 90 degrees for any normalization. The situation can be clarified by normalization for $c_n = 1$, as done by Lipshitz and Vanderkooy in [1] and [5], where $c_0 = 1$ for unity gain at 0 Hz. This form reveals any inherent asymmetry. Equation (6) shows the general low-pass, while (7) is the fourth-order Bessel denominator. Note that it becomes nearly symmetric, and relatively similar to the Linkwitz-Riley in (4).

$$\frac{1}{c_o + c_1 \cdot s + c_2 \cdot s^2 + \dots + c_n \cdot s^n} \xrightarrow[s \to \frac{s}{\sqrt[n]{c_n}}]{c_o = 1} \frac{1}{1 + k_1 \cdot s + k_2 \cdot s^2 + \dots + s^n} \qquad (6)$$

$$1 + s + \frac{9}{21} \cdot s^2 + \frac{2}{21} \cdot s^3 + \frac{1}{105} \cdot s^4 \longrightarrow 1 + 3.2011 \cdot s + 4.3916 \cdot s^2 + 3.1239 \cdot s^3 + s^4 \qquad (7)$$

PHASE-MATCHED BESSELS

The textbook low-pass Bessel is often designed for an approximate time delay of $1/\omega_o$, rather than for the common -3 dB or -6 dB level at ω_o used for crossovers. This design will be used as a reference, to which other normalizations are compared. It has quite a lot of overlap, with very little attenuation at the design frequency, as shown in Fig. 3, for a second-order Bessel with the typical reversed connection.

Bessel polynomials of degree three or higher are not inherently symmetric, but may be normalized to be nearly symmetric by requiring a phase shift at ω_o of 45 degrees per order, negative for the low-pass, positive for the high-pass. This results in a fairly constant relative phase at all other frequencies. Equation (8) shows an equation for deriving the normalization constant of the fourth-order Bessel, where the imaginary part of the denominator is set to zero for 180-degree phase shift at ω_p.

$$\omega_p - \frac{2}{21}\cdot\omega_p^{\,3} = 0 \;, \quad \omega_o = \frac{1}{\omega_p} = \frac{1}{\sqrt{10.5}} \qquad (8)$$

This normalization is not new, but was presented in a slightly different context in [5], with a normalization constant of 0.9759, which is the square of the ratio of the phase-match ω_o in equation (8) to the ω_o in equations (6) and (7).

Since the phase nonlinearity of the high-pass is now in the passband, the crossover resulting from the sum of the two approaches phase linearity only at lower frequencies. This doesn’t preclude it from being a useful crossover.

The summed magnitude response of the Bessel normalized by the 45-degree criterion is fairly flat, within 2dB for the second-order and fourth-order. We may adjust the overlap slightly for flattest magnitude response instead, at the expense of the polar response. Figures 4-6 show the results of four normalizations for the second-order filter. The –3 dB and phase-match normalizations are illustrated in Fig. 5 and 6. Note that for the second-order phase-match design, low-pass and high-pass group delays are exactly the same.

The fourth-order is illustrated in Fig. 7-9, Fig. 7 being a 3-D plot of frequency response versus normalization. Fig. 8 shows four cases, which are cross-sections of figure 7. The phase-match case has good flatness as well as the best polar response. The fourth-order Linkwitz-Riley is very similar when viewed about $\omega_o = 1.0$, compared to 0.31 on Fig. 7. The third-order Bessel magnitude has comparable behavior.

In a real application, phase shifts and amplitude variations in the drivers will require some adjustment of the overlap for best performance. The sensitivity of the crossover response to normalization should be considered [2].

COMPARISON OF TYPES

Butterworth, Linkwitz-Riley, and Bessel crossovers may be thought of as very separate types, while in fact they are all particular cases in a continuous space of possible crossovers. The separate and summed magnitude responses are distinct but comparable, as can be seen by graphing them together (Fig. 10). The Bessel and Linkwitz-Riley are the most similar. The Butterworth has the sharpest initial cutoff, and a +3dB sum at crossover. The Linkwitz-Riley has moderate rolloff and a flat sum. The Bessel has the widest, most gradual crossover region, and a gentle dip in the summed response. All responses converge at frequencies far from ω_o .

The phase responses also look similar, but the amount of peaking in the group delay curve varies somewhat, as shown in Fig. 11. There is no peaking in the Bessel low-pass, while there is a little in the high-pass for orders > 2. The summed response has only a little peaking. The group delay curve is directly related to the behavior in the time domain, as discussed in [11]. The most overshoot and ringing is exhibited by the Butterworth design, and the least by the Bessel.

Often when discussing crossovers, the low-pass step response is considered by itself, while the high-pass and summed step response is usually far from ideal, except in the case of the linear phase crossover; this has been known for some time [12], but step-response graphs of higher-order crossovers are generally avoided out of good taste!

Table 1 gives Bessel crossover denominators normalized for time delay and phase match. Note the relative symmetry for the phase-match cases.

In summary, it is seen that a Bessel crossover designed as described above is not radically different from other common types, particularly compared to the Linkwitz-Riley. It does not maintain linear phase response at higher frequencies, but has the most linear phase of the three discussed, along with fairly good magnitude flatness and minimal lobing for the even orders. It is one good choice when the drivers used have a wide enough range to support the wider crossover region, and when good transient behavior is desired.

ACKNOWLEDGMENTS

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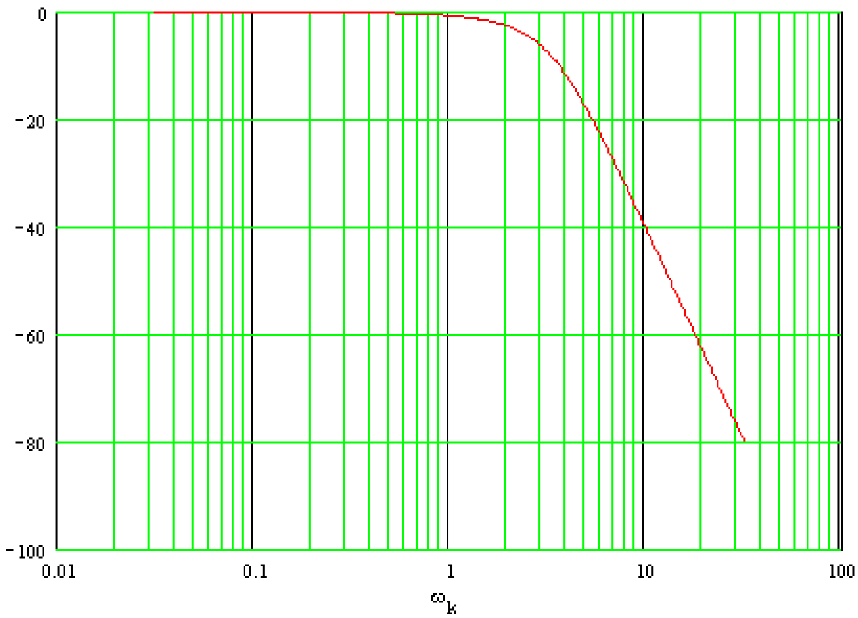

Fig. 1 - Fourth-Order Bessel Magnitude

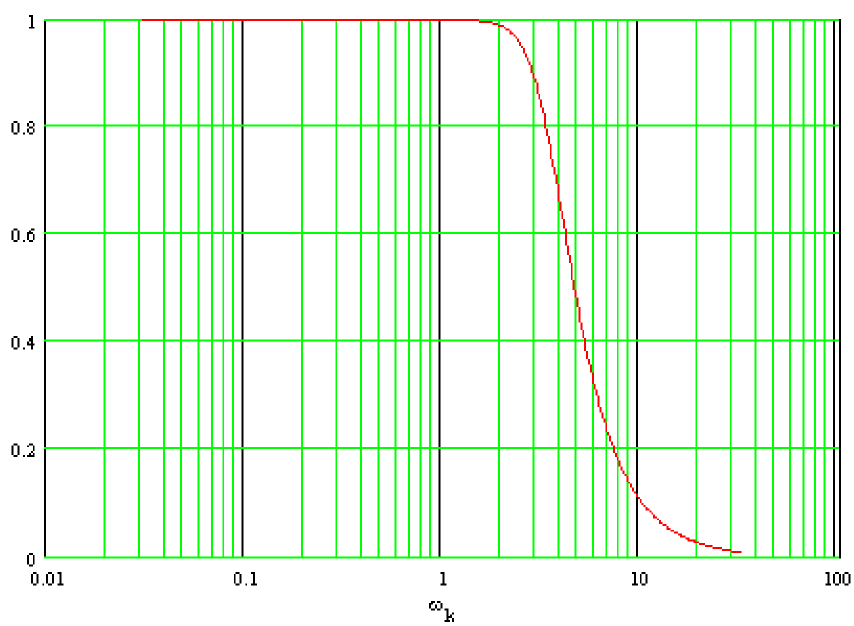

Fig. 2 - Fourth-Order Bessel Group Delay

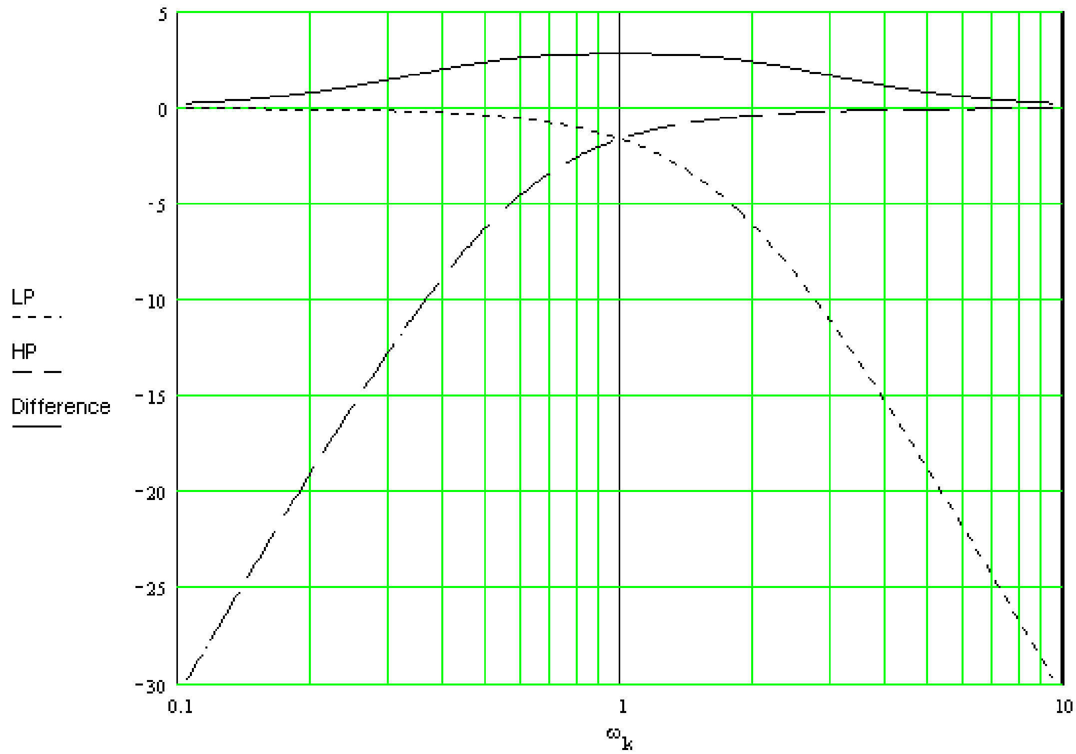

Fig. 3 - Second-Order Bessel Crossover

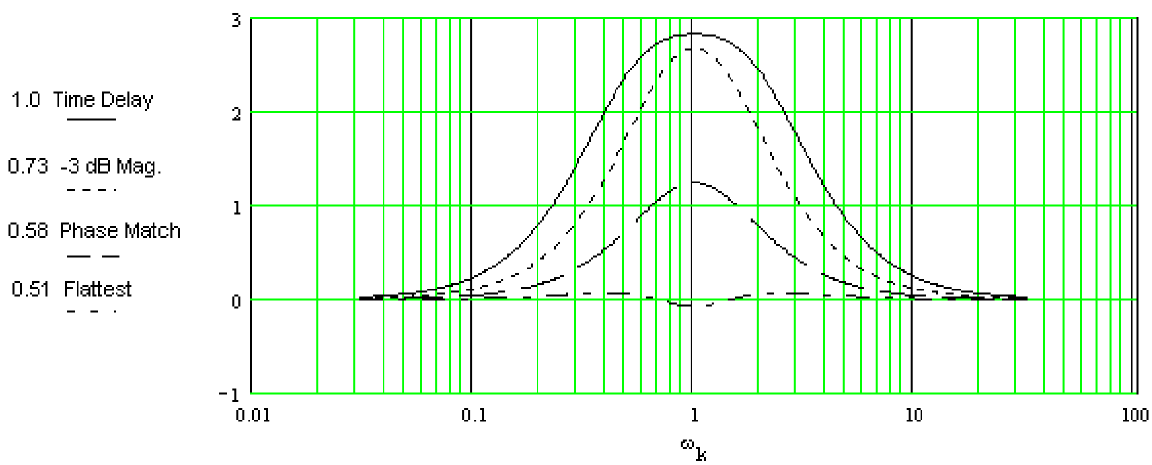

Fig. 4 - Comparison of Second-Order Bessel Sums

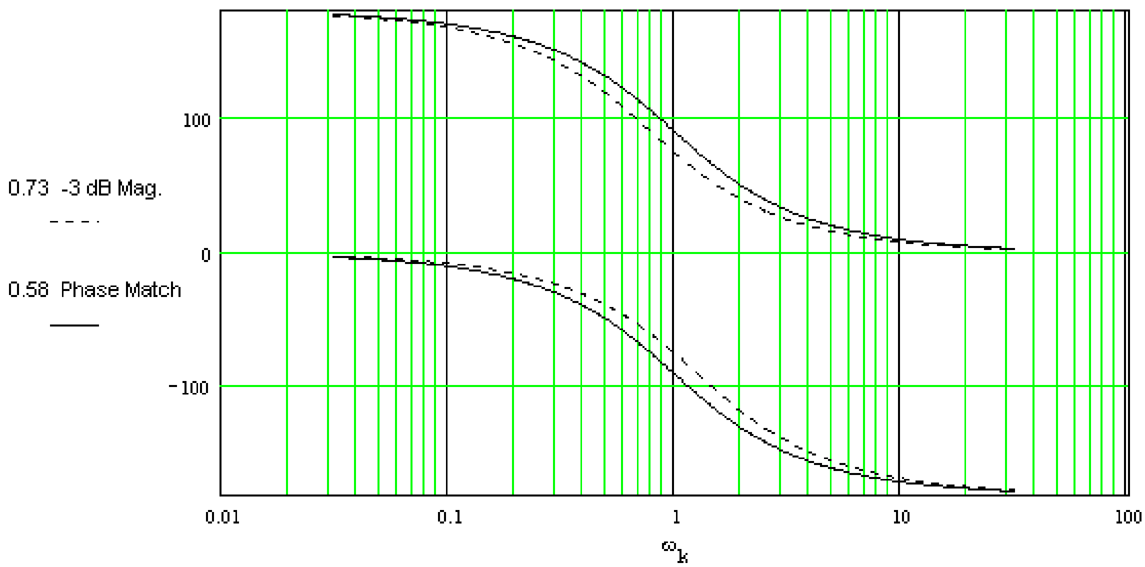

Fig. 5 - Comparison of Second-Order Phase

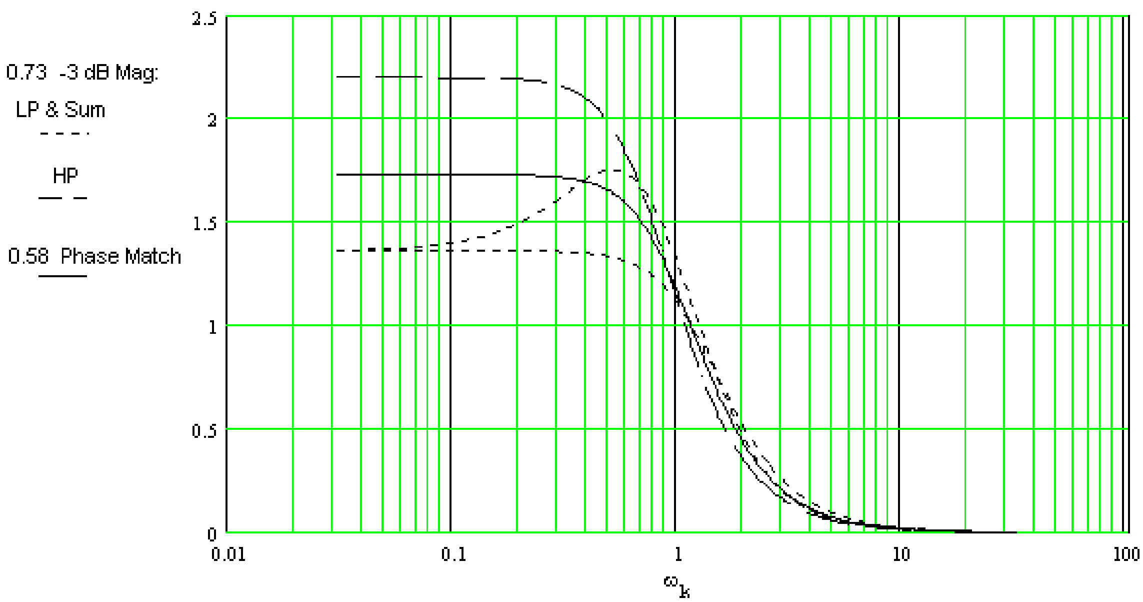

Fig. 6 - Second-Order Group Delay

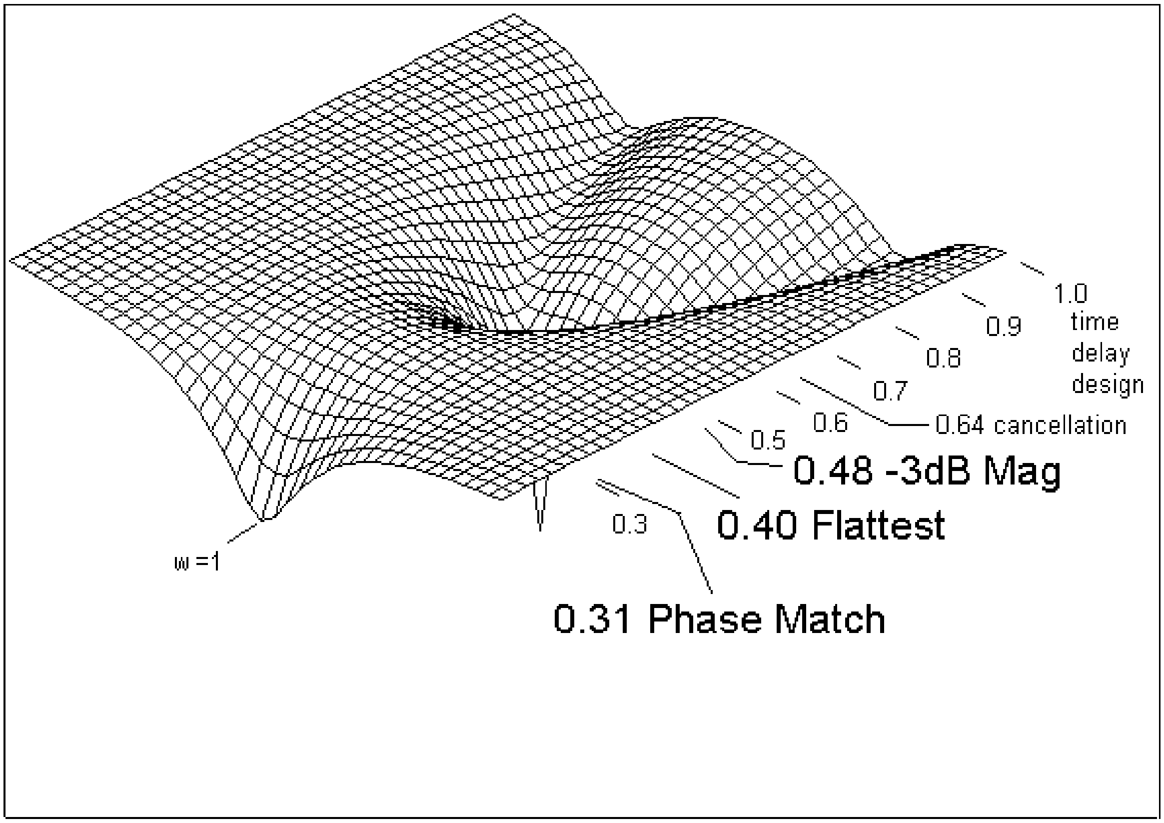

Fig. 7 - Summed Fourth-Order Bessel Frequency Response vs. Normalization
Normalization values are relative to time-delay design.

Summed Fourth-Order Bessel Frequency Response versus Overlap-Normalization, Relative to Time-Delay Design

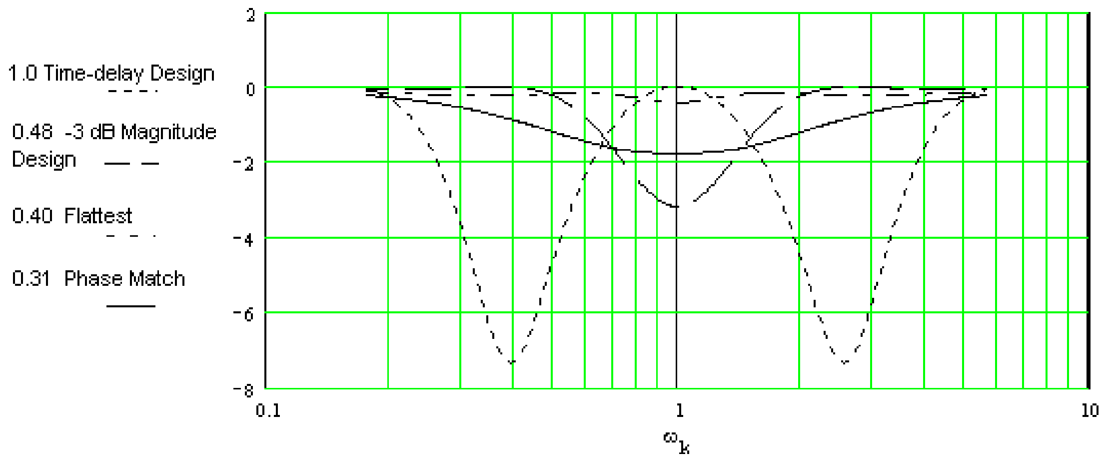

Fig. 8 - Summed Fourth-Order Responses

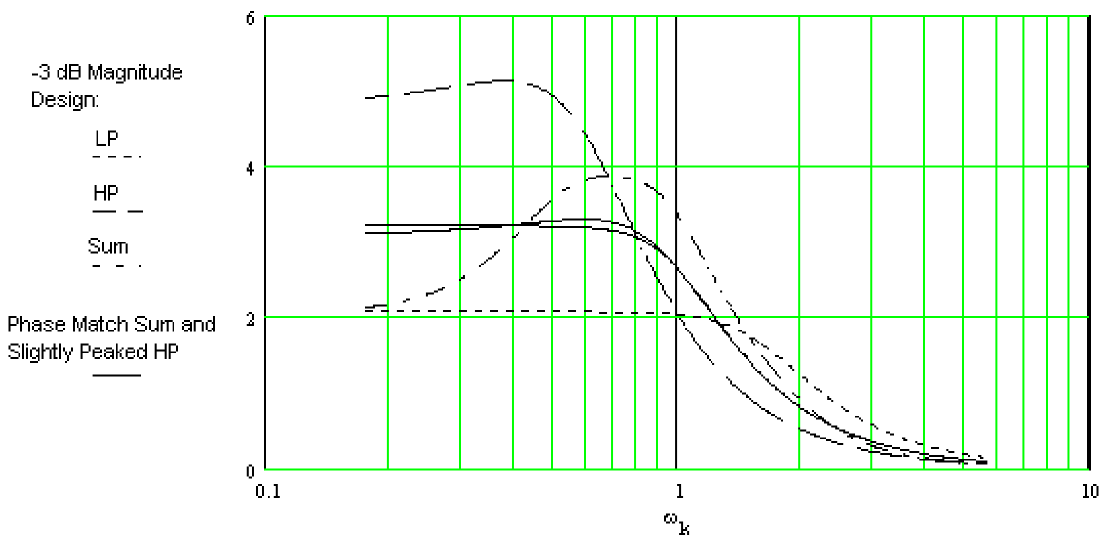

Fig. 9 - Fourth-Order Sum Group Delays

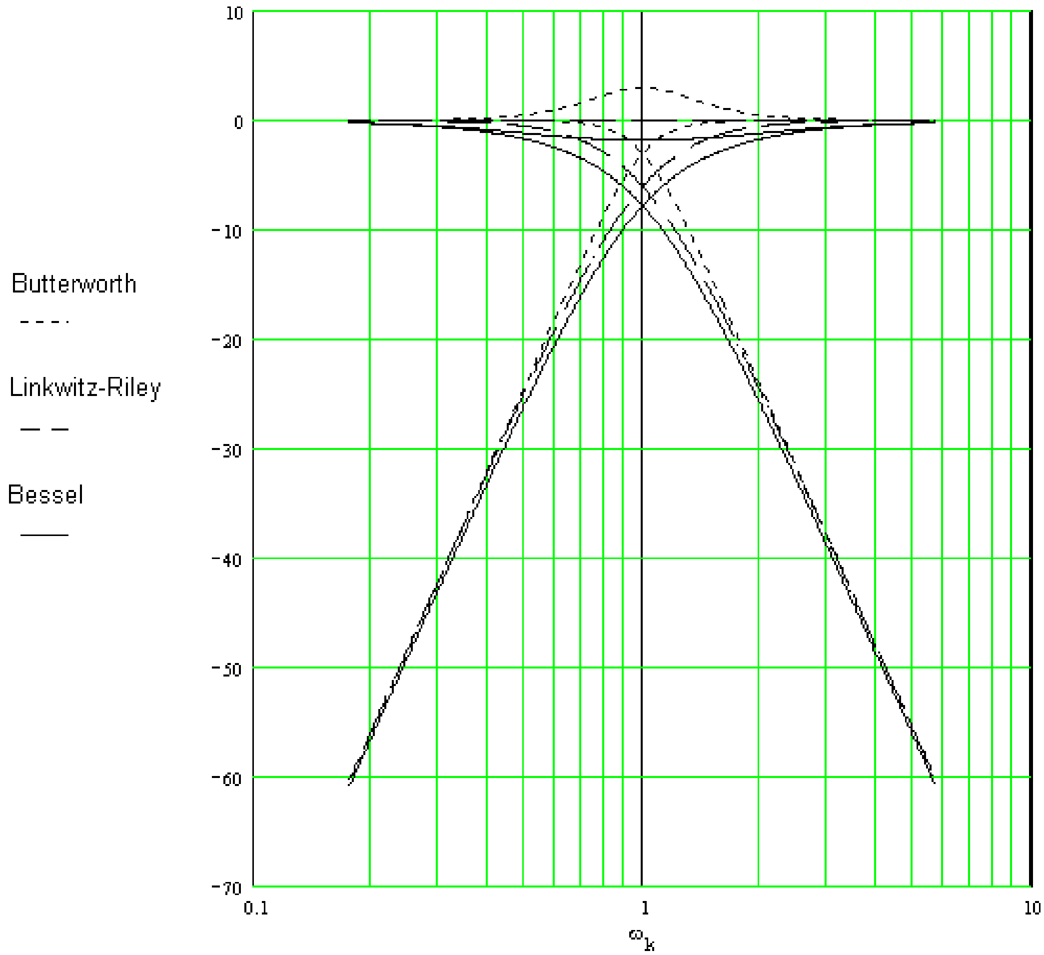

Fig. 10 - Fourth-Order Magnitudes

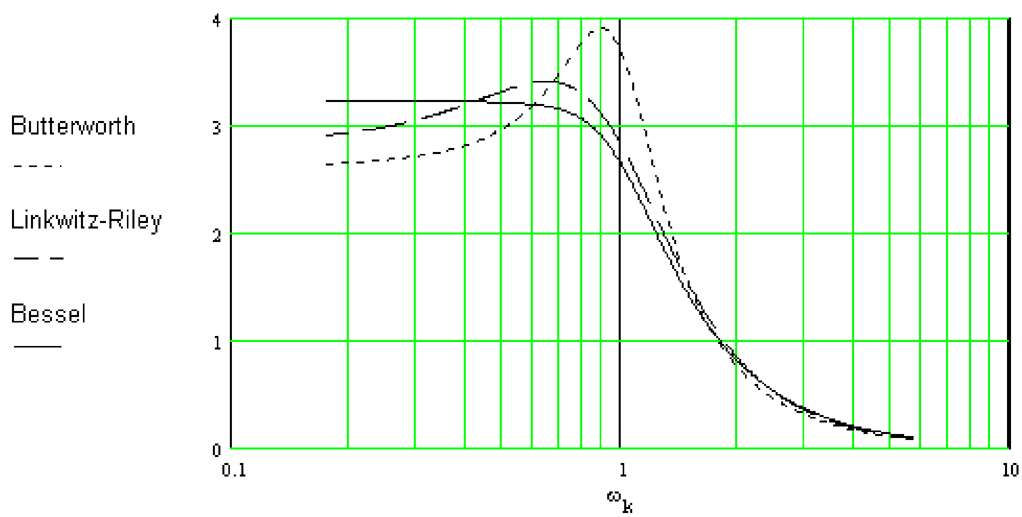

Fig. 11 - Fourth-Order Group Delays